Optimization
A Journal of Mathematical Programming and Operations Research

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gopt20

Confidence regions of stochastic variational inequalities: error bound approach

Yongchao Liu & Jin Zhang

OPTIMIZATION
https://doi.org/10.1080/02331934.2020.1857755

Confidence regions of stochastic variational inequalities: error bound approach

Yongchao Liu[a] and Jin Zhang[b]

[a]School of Mathematical Sciences, Dalian University of Technology, Dalian, People's Republic of China; [b]Department of Mathematics, SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, People's Republic of China

ABSTRACT

In this paper, we aim to build confidence regions of the true solution to the stochastic variational inequalities problem (SVIP) when the sample average approximation (SAA) scheme is implemented. A new approach based on error bound conditions admitted by the SVIP is proposed. This so-called *error bound approach* provides an upper bound of the distance between SAA solutions and the true solution set through the distance between the SAA function and the true counterpart at the SAA solutions. Certain statistical tools such as central limit theorem and Owen's empirical likelihood theorem are then employed to construct the asymptotic confidence regions of the solutions to SVIP. In particular, if the SVIP admits a global error bound condition, the non-asymptotic (uniform) confidence regions of the solutions are also approachable. Different from the conventional normal map approach, our error bound approach does not require any information regarding the derivative of the solution mapping with respect to perturbations of involved functions in SVIP. For constructing component-wise confidence regions, the validity of the error bound approach is guaranteed for those cases where the functions own separable structures.

ARTICLE HISTORY
Received 3 February 2020
Accepted 13 November 2020

KEYWORDS
Stochastic programming; stochastic variational inequalities; (non-)asymptotic confidence regions; empirical likelihood method

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
90C33; 90C15

1. Introduction

The variational inequalities problem (VIP): given a subset $\mathcal{C}$ of the Euclidean space $\mathbb{R}^n$ and a mapping $F : \mathcal{C} \to \mathbb{R}^n$, the variational inequality is to find a vector $x \in \mathcal{C}$ such that

$$(y - x)^T F(x) \geq 0, \quad \forall\, y \in \mathcal{C}.$$

VIP has many applications in engineering, economics, game theory and networks and has been extensively studied in the past decades, see, e.g. [1]. In order to describe decision making problems which involve future uncertainty, the stochastic version of variational inequalities problem (SVIP) was proposed.

CONTACT Yongchao Liu lyc@dlut.edu.cn

Different approaches to incorporate the uncertainty into VIP induce different SVIP models.

ERM-SVIP model: Chen and Fukushima [2] extend VIP to SVIP as: find $x \in \mathcal{C}$ such that

$$(y - x)^T F(x, \xi) \geq 0, \quad \forall\, y \in \mathcal{C}, \quad \forall\, \xi \in \Xi, \tag{1}$$

where $\mathcal{C} \in \mathbb{R}^n$ and $\xi : \Omega \to \Xi$ is a vector of random variables defined on probability $(\Omega, \mathcal{F}, P)$ with support set $\Xi \subset \mathbb{R}^m$. Obviously, such x satisfying (1) hardly exists. To address this concern, the authors suggest finding solutions to the following expected residual minimization (ERM) problem

$$\min_{x \in \mathcal{C}} \mathbb{E}_P[\theta(x, \xi)], \tag{2}$$

where $\mathbb{E}_P[\cdot]$ denotes the expected value with respect to the distribution P of ξ and $\theta(\cdot)$ is a residual function which satisfies $\theta(x, \xi) \geq 0$ for any $(x, \xi) \in \mathcal{C} \times \Xi$ and $\theta(x, \xi) = 0$ if and only if x solves (1). Models (1)–(2) are known as ERM-SVIP in the literature.

EV-SVIP model: Another way to formulate SVIP is proposed by Gürkan et al. [3] which is a natural extension of deterministic VIP: find $x \in \mathcal{C}$ such that

$$(y - x)^T f(x) \geq 0, \quad \forall\, y \in \mathcal{C}, \tag{3}$$

where $f(x)$ denotes the expected value (EV) of $F(x, \xi)$ with respect to the distribution P of ξ, i.e.

$$f(x) := \mathbb{E}_P[F(x, \xi)]. \tag{4}$$

$\mathcal{L}^p$-SVIP model: We may also consider the SVIP in the functional space [4]: for almost every $\xi \in \Xi$, find $x \in \mathcal{C}$ such that

$$(y - x)^T F(x, \xi) \geq 0, \quad \forall\, y \in \mathcal{C}. \tag{5}$$

Different with the ERM-SVIP and EV-SVIP, the solution to $\mathcal{L}^p$-SVIP is a measurable function as it depends on the random variable ξ. Moreover, the set $\mathcal{C}$ in $\mathcal{L}^p$-SVIP can be dependent on the random variable ξ, that is, $\mathcal{C} : \Xi \rightrightarrows \mathbb{R}^n$ is a measurable set-valued mapping. As the paper focuses on the statistical properties of the sample average approximation method based solutions to SVIP, we only discuss the ERM-SVIP and EV-SVIP in next. Please see [5–9] for the new theory, algorithm and application of $\mathcal{L}^p$-SVIP model.

For both ERM-SVIP model (1)–(2) and EV-SVIP model (3), the essential difficulty in designing numerical methods is associated with computing the expected value $\mathbb{E}_P[\cdot]$. In fact, if $\mathbb{E}_P[\theta(x, \xi)]$ admits a closed form representation, ERM-SVIP model (1)–(2) turns out to be a deterministic optimization problem. Similar is the case EV-SVIP model (3). However, in practice, obtaining a closed form of $\mathbb{E}_P[\cdot]$ or computing the value of it numerically is usually difficult either due to the unavailability of the distribution of ξ or because it involves multiple integration. Instead, it is more realistic to obtain a sample of the random vector ξ either from past data or from computer simulation. Consequently, one may consider approximating the SVIP based on sampling $\xi_1, \dots, \xi_N$. Specifically, approximations based on sampling $\xi_1, \dots, \xi_N$ for ERM-SVIP model (1)–(2) and EV-SVIP model (3) read as

$$\theta_N(x) := \frac{1}{N} \sum_{j=1}^{N} \theta(x, \xi_j) \quad \text{and} \quad f_N(x) := \frac{1}{N} \sum_{j=1}^{N} F(x, \xi_j), \tag{6}$$

respectively. This kind of approximation technique is well-known in stochastic programming under various names such as sample average approximation (SAA), Monte Carlo method, sample path optimization etc.

The SAA technique suggests approximating ERM-SVIP (1)–(2) as:

$$\min_{x \in \mathcal{C}} \theta_N(x). \tag{7}$$

It also leads to the approximation of EV-SVIP (3) as: find $x \in \mathcal{C}$ such that

$$(\text{SAA} - \text{SVIP}) \quad (y - x)^T f_N(x) \geq 0, \quad \forall\, y \in \mathcal{C}. \tag{8}$$

A natural question to be answered is how well SAA solutions approximate true solutions. One popular way is to study the consistence of SAA solutions. For ERM-SVIP model (1)–(2), Chen and Fukushima [2] prove that the solutions generated by the SAA approximation converge to the true solution as N tends to infinity. For SVIP model (3), Gürkan et al. [3] propose simulation based sample-path optimization approach for solving SVIP (1) and show the consistence of the solutions. Indeed, the study on the consistence of SAA solutions of SVIP can be traced back to the works by King and Rockafellar [10,11] where the authors study asymptotic convergence and statistical properties of the stochastic generalized equation (SGE).

Instead of studying the asymptotic convergence, very recently, [12–17] focus on the inference, i.e. the construction of confidence regions of solutions to EV-SVIP (3). They initiate methods to compute confidence regions for the true solution from a single SAA solution, which they call *normal map approach* given that the method is based on normal map $F^{\text{nor}}_{\mathcal{C}}(z)$, see Section 2.1 for the definition. It is easy to see that x is a solution to SVIP (3) if and only if there exists a vector z such that $x = \Pi_{\mathcal{C}}(z)$ and $F^{\text{nor}}_{\mathcal{C}}(z) = 0$, where $\Pi_{\mathcal{C}}(\cdot)$ denotes the Euclidean projection onto $\mathcal{C}$. The idea behind the normal map approach is to build the confidence region of solution to $F^{\text{nor}}_{\mathcal{C}}(z) = 0$, therefore the confidence region of solution to SVIP (3) can be further induced by projection. Under certain assumptions, e.g. the uniqueness and Lipschitz continuity of the solutions mapping $z(f)$[1] of $F^{\text{nor}}_{\mathcal{C}}(z) = 0$, thanks to Delta method, the difference between the SAA solutions and the true solution converges to a normal distribution. Moreover, the rate of such convergence depends on the differential of the normal map at the true solution. As the true solution to SVIP (3) is usually unknown in practice, efforts have been devoted to approximating the differential of normal map through SAA solutions. However, due to lack of continuity, using SAA counterpart as an estimation of differential of normal map is to some extent too ambitious. One of the main objectives of normal map approach is to overcome such difficulty.

In this paper, we revisit issues concerning the confidence regions of solutions to SVIP (3). Instead of the normal map approach studied in [12–17], we propose the so-called *error bound approach*. In particular, we shall follow the following roadmap to build the confidence region of the true solution in terms of certain error bound conditions.

Step 1. We shall provide an upper bound estimation of the distance between the SAA solutions and the true solution set through distance between SAA residual functions and original residual function at the SAA solution.

Step 2. We next study the statistical properties of the random vector[2] through the statistical tools such as central limit theorem (CLT) and Owen's empirical likelihood theorem (ELT) [18]. As the error bound approach does not rely on Delta method, information of the differential of normal map is no long required. In this way, we may avoid assuming uniqueness of the solution to SVIP (3).

Step 3. Furthermore, under certain error bound conditions, we may construct the non-asymptotic confidence regions of the solutions to SVIP (3) by the large deviation theorem (LDT). As far as we know, this is the first result on the non-asymptotic confidence regions of the solutions to SVIP (3), which may shed some light on collecting samples.

More specifically, we may summarize the main structure as follows:

$$\mathrm{dist}(x_N, S) \underset{\leq}{\xrightarrow{\text{EB}}} \mathrm{d}\,(f_N(x_N), f(x_N)) \begin{cases} \xrightarrow{\text{CLT}} & \text{AsymptoticCF}(N(\mu,\sigma^2)) \\ \xrightarrow{\text{Owen's ELT}} & \text{AsymptoticCF}(\chi^2(d)) \\ \xrightarrow{\text{LDT}} & \text{Non}-\text{asymptoticCF} \end{cases} \tag{9}$$

where $\mathrm{dist}(x_N, S)$ denotes the distance between the solution x_N to SAA-SVIP (8) and the set of solutions S of true SVIP (3), 'EB' is short for error bound conditions, CF is short for confidence regions, and $\mathrm{d}(f_N(x_N), f(x_N))$ means distance determined by the SAA function $f_N(\cdot)$ (6) and the true counterpart $f(\cdot)$ (4) at point x_N. The definition of $\mathrm{dist}(\cdot,\cdot)$ will be more specified when the type of error bound conditions is determined.

The organization of this paper is as follows. Section 2 reviews the normal map approach [16] and error bound conditions of VIP. The theoretical illustration for diagram (9) are presented in Section 3 with CLT based normal distribution in Subsection 3.1, Owen's ELT in Subsection 3.2 and non-asymptotic results in Subsection 3.3 respectively. Section 4 presents some numerical results on the confidence regions of SVIP (3).

2. Preliminaries

In the first part of this section, we review the normal map approach discussed in [12–17] and outline its main idea. In the second part, we introduce some preliminary results on error bound conditions of VIP, which will be used to construct the confidence regions of the solutions to SVIP (3) in Section 3.

2.1. Normal map approach

The normal map induced by function $f(\cdot)$ and convex set $\mathcal{C}$ reads as:

$$F_{\mathcal{C}}^{\rm nor}(z) := f(\Pi_{\mathcal{C}}(z)) + z - \Pi_{\mathcal{C}}(z).$$

One of the key conditions of the normal map approach [12–17] is the Lipschitz continuity and differentiability of the solution mapping to $F_{\mathcal{C}}^{\rm nor}(z) = 0$. To address this concern, the following two assumptions are required.

Assumption 2.1 ([16, Assumption 3.1]):

(a) $\mathbb{E}[\|F(x,\xi)\|^2] < \infty$ for all $x \in \mathcal{C}$.
(b) The map $x \to F(x,\xi)$ is continuously differentiable on $\mathcal{C}$ for almost every $\xi \in \Xi$.
(c) There exists a square integrable random variable $c(\cdot)$ such that

$$\|F(x,\xi) - F(x',\xi)\| + \|d_xF(x,\xi) - d_xF(x',\xi)\| \le c(\xi)\|x - x'\|$$

for almost every $\xi \in \Xi$.

Assumption 2.1 is the standard condition which ensures the uniform convergence of SAA function $f_N(x)$ toward $f(x)$ and $d_xf_N(x)$ toward $d_xf(x)$. If we concentrate on the convergence of $f_N(\cdot)$ toward $f(\cdot)$, actually the continuously differentiable condition in (b) can be replaced by Lipschitz continuity. Therefore, the corresponding norm $\|d_xF(x,\xi) - d_xF(x',\xi)\|$ in (c) can be removed.

Assumption 2.2 ([16, Assumption 3.2]): Suppose that x_0 solves SVIP (3). Let $z_0 = x_0 - f(x_0)$, $L = df(x_0)(\cdot)^3$, $\mathcal{K}_0 = T_{\mathcal{C}}(x_0) \cap \{z_0 - x_0\}^{\perp}$ and assume that the normal map $L_{\mathcal{K}_0}^{nor}(\cdot)$ is a homeomorphism from $\mathbb{R}^n \to \mathbb{R}^n$, where $T_{\mathcal{C}}(x_0)$ is the tangent cone of $\mathcal{C}$ at point x_0 and $L_{\mathcal{K}_0}^{nor}(\cdot)$ is the normal map induced by L and $\mathcal{K}_0$.

Assumptions 2.1–2.2 together ensure that SVIP (3) has a locally unique solution under sufficiently small perturbation on $f(\cdot)$. Moreover, the solution mapping is indeed Lipschitz continuous and differentiable with respect to perturbations of function $f(\cdot)$.

Lemma 2.1 ([16, Lemma 3.1]): *Under Assumptions 2.1 and 2.2, the normal map $L_{\mathcal{K}_0}^{nor}(\cdot)$ has a positive injectivity modulus δ on $\mathbb{R}^n$. Moreover, for each $\lambda > \delta^{-1}$*

there exist neighbourhoods $\mathcal{N}(x_0)$ of x_0 in $\mathcal{C}$, $\mathcal{N}(z_0)$ of z_0 in $\mathbb{R}^n$ and $\mathcal{N}(f)$ of f in $C^1(\mathcal{C}, \mathbb{R}^n)$[4]*, and a function $z : \mathcal{N}(f) \to \mathbb{R}^n$, such that:*

(i) $z(f) = z_0$.
(ii) *For each $g \in \mathcal{N}(f)$, $z(g)$ is the unique point in $\mathcal{N}(z_0)$ satisfying $G_{\mathcal{C}}^{\mathrm{nor}}(z(g)) = 0$, where $G_{\mathcal{C}}^{\mathrm{nor}}(z)$ is the normal map induced by function $g(\cdot)$ and convex set $\mathcal{C}$, and $x(g) = \Pi_{\mathcal{C}}(z(g))$ is the unique point in $\mathcal{N}(x_0)$ satisfying $0 \in f(x(g)) + \mathcal{N}_{\mathcal{C}}(x(g))$.*
(iii) *$z(\cdot)$ is Lipschitz on $\mathcal{N}(f)$ with modulus λ.*

Finally, the functions $z(\cdot)$ and $x(\cdot)$ are B-differentiable at f.

Upon combining Lemma 2.1 and the Delta method, the convergence in distributions of SAA solutions to the solutions of SVIP (3) can be achieved.

Lemma 2.2 ([16, Theorem 5.1]): *Suppose that Assumptions 2.1 and 2.2 hold. Let $z_N \to z_0$ and $x_N \to x_0$. Then*

$$\sqrt{N}(z_N - z_0) \Rightarrow (L_{\mathcal{K}_0}^{nor})^{-1}(Y_0),$$

where Y_0 is a normal random vector with zero mean and covariance matrix Σ_0 which depends on the true solution z_0, and '$\Rightarrow$' denotes the convergence in distribution.

Lemma 2.2 can be used to build the confidence region of the solutions to SVIP (3). However, the limit distribution depends on the true solution x_0 through covariance matrix Σ_0 and $L_{\mathcal{K}_0}^{nor}$ which are typically unknown. Most efforts in line of this analysis have been devoted to approximating unknown parameters through their SAA counterparts [12–17].

2.2. Error bound conditions

Error bound conditions of VIP estimate the unknown distance from a given point to the solution set in terms of an easily computable residual. Error bound theory plays a key role in subdifferential calculus rules, optimality condition, stability analysis, algorithmic convergence and etc. In this subsection, we recall some widely known error bound conditions of VIP presented in [1, Chapter 6]. We refer readers interested in this topic to a survey paper [19] and the reference therein.

Definition 2.1: Consider the deterministic VIP:

$$(y - x)^T f(x) \geq 0, \quad \forall\, y \in \mathcal{C}, \tag{10}$$

and denote the set of solutions to (10) as $S(f, \mathcal{C})$. VIP (10) is said to admit a local error bound at point $x^* \in S(f, \mathcal{C})$ if there exists a neighbourhood $\mathcal{N}(x^*)$ of x^*, two positive constants c_1 and c_2, and a residual function $r : \mathcal{N}(x^*) \to \mathbb{R}_+$ such that

$$\mathrm{dist}(x, S(f, \mathcal{C})) \begin{cases} \leq c_1 r(x)^{c_2}, & \forall\, x \in \mathcal{N}(x^*), \\ = 0, & \forall x \in S(f, \mathcal{C}) \cap \mathcal{N}(x^*), \end{cases} \tag{11}$$

where $\mathrm{dist}(\bar{a}, A) := \inf_{a \in A} \|\bar{a} - a\|$ for given point $\bar{a}$ and set A in $\mathbb{R}^n$.

We say that a Lipschitzian error bound condition holds for VIP (10) if the inequality in (11) holds with $c_2 = 1$. For the case that (11) holds with $c_2 \neq 1$, we say that VIP (10) admits a Hölderian error bound condition. If the neighbourhood $\mathcal{N}(x^*)$ is replaced by $\mathbb{R}^n$, we say that a global error bound condition holds. In what follows, we recall some error bound conditions which have been intensively studied in the literature.

The first example shows that Assumptions 2.1–2.2 suffice to ensure a certain type error bound, namely, the normal map error bound condition.

Example 2.1 (Normal map error bound condition): VIP (10) admits the *normal map error bound condition* if

$$\mathrm{dist}(x, S(f, \mathcal{C})) \leq c \|F_{\mathcal{C}}^{\mathrm{nor}}(x)\|, \tag{12}$$

where $F_{\mathcal{C}}^{\mathrm{nor}}(\cdot)$ is the normal map induced by $f(\cdot)$ and $\mathcal{C}$. As presented in Lemmas 2.1–2.2, normal map lies at the heart of the normal map approach. Under Assumptions 2.1–2.2, Lemma 2.1 states that the solution mapping is unique and Lipschitz continuous with respect to the perturbations of function $f(\cdot)$. Obviously, Lipschitz continuity of solution mapping implies the semi-stability of SVIP (3), that is, for every open U containing $S(f, \mathcal{C})$, there exist two positive scalars γ_1 and γ_2 such that, for every function in the γ_2 ball of function f[5], it holds that

$$S(g, \mathcal{C}) \cap U \subseteq S(f, \mathcal{C}) + \gamma_1 \mathcal{B},$$

where $S(g, \mathcal{C})$ denotes the solution set of VIP and $\mathcal{B}$ denotes the unit ball of $\mathbb{R}^n$:

$$(y - x)^T g(x) \geq 0, \quad \forall\, y \in \mathcal{C}.$$

(see Definition 5.5.1 of [1] for details). According to [1, Proposition 5.5.5], the normal map error bound condition holds.

The second and third examples are two well-known error bound conditions regarding complementarity problems.

Example 2.2 (Natural type error bound condition): Consider the following complementarity problem

$$0 \leq g(x) \perp h(x) \geq 0, \tag{13}$$

where $g(\cdot)$ and $h(\cdot)$ are continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. We say that complementarity problem (13) admits the *natural error bound condition* if there exist positive constants c and δ such that for any $x \in \mathcal{B}(S(g,h),\delta)$

$$\operatorname{dist}(x, S(g,h)) \leq c \left\| \min\{g(x), h(x)\} \right\|, \tag{14}$$

where $S(g,h)$ denotes the set of solutions to (13) and $\mathcal{B}(S(g,h),\delta)$ denotes the δ-neighbourhood of set $S(g,h)$. If $g(x) := x$ and $h(x)$ is a Lipschitz continuous, uniform P-function with Lipschitz modulus L and constant γ, i.e. there exists constant $\gamma > 0$,

$$\max_{1\leq i\leq m} [h_i(x) - h_i(y)](x_i - y_i) \geq \gamma \|x - y\|^2 \quad \forall\, x, y \in \mathbb{R}^n,$$

then the natural error bound condition (14) holds with $c = \frac{1+L}{\gamma}$. Moreover, the natural error bound condition (14) is indeed in a global manner, i.e.

$$\operatorname{dist}(x, S(g,h)) \leq \frac{1+L}{\gamma} \left\| \min\{g(x), h(x)\} \right\|, \quad \forall\, x \in \mathbb{R}^n.$$

Example 2.3 (S-type error bound condition): The complementarity problem (13) is said to admit the *S-type error bound condition* if there exist positive constants c and δ such that for $x \in \mathcal{B}(S(g,h),\delta)$

$$\operatorname{dist}(x, S(g,h)) \leq c \left\| (-g(x), -h(x), g(x) \circ h(x))_+ \right\|, \tag{15}$$

where $(a)_+ := \max\{a, 0\}$ for a vector 'a' and the maximum is taken componentwise and '$\circ$' denotes the Hadamard product. S-type error bound condition is often related to monotone complementary problems. In particular, for linear complementarity problems, e.g. $g(x) := x$ and $h(x) := Ax + q$, if A is a semi-definite matrix and

$$0 \leq x \perp h(x) \geq 0 \tag{16}$$

has a non-degenerate solution[6], then there exists a constant $c > 0$ such that the S-type error bound condition (15) holds. Furthermore, without the non-degenerate condition, the complementarity problem (13) admits a weaker S-type error bound condition as follows:

$$\begin{aligned} &\operatorname{dist}(x, S(g,h)) \\ &\quad \leq c\left(\left\| (-g(x), -h(x), g(x) \circ h(x))_+ \right\| + \sqrt{\left\| (-g(x), -h(x), g(x) \circ h(x))_+ \right\|} \right). \end{aligned} \tag{17}$$

3. Error bound approach

The normal map approach relies heavily on approximating directional derivative of the solution mapping through the SAA counterpart, see, e.g. [12–17]. This task sometimes turns out to be not easy as the directional derivative may be lack of continuity. This observation motivates us to employ the error bound condition to build the confidence region of the solution to SVIP (3). In fact, the norm of the derivative of the solution mapping can be regarded as the local Lipschitz constant of the mapping, which characterizes the variety of the solution mapping with respect to the variety functions. Error bound condition, on the other hand, performs in a similar way. In comparison, the error bound condition characterizes the variety of the solution mapping in a more general manner as it allows different types of residual functions. Moreover, the error bound of the VIP has been well studied in the past decades, see, e.g. Chapter 6 of [1] and survey paper [19].

As depicted in diagram (9), we shall first provide an upper bound of the distance between SAA solutions and the true one in terms of error bound conditions.

Theorem 3.1: *Let $\{x_N\}$ be a sequence of solutions to SAA-SVIP (8) and S be the set of solutions to SVIP (3). Suppose that the error bound condition (11) holds with residual function $r(\cdot)$. Suppose further that the corresponding residual function $r_N(\cdot)$[7] for SAA-SVIP (8) converges to $r(\cdot)$ uniformly on $\mathcal{C}$ with probability one (w.p.1). Then*

(i) *the limit point x^* of sequence $\{x_N\}$ is a solution to SVIP (3) w.p.1;*
(ii) *there exists a sufficiently large N' such that, for any $N \geq N'$,*

$$dist(x_N, S) \leq c_1(r(x_N) - r_N(x_N))^{c_2},$$

w.p.1.

Proof: (i). Denote x^* as the limit point of sequence $\{x_N\}$. By the assumption that $r_N(\cdot)$ converges to $r(\cdot)$ on $\mathcal{C}$ uniformly, $r(x^*) = 0$ as $r_N(x_N) = 0$ and $x_N \to x^*$. Then x^* is a solution to SVIP (3) w.p.1.

(ii). Following part (i) and taking a subsequence if necessary, there exists a sufficiently large N' such that $x_N \in \mathcal{B}(S, \delta)$, where $\mathcal{B}(S, \delta)$ denotes the δ-neighbourhood of S. According to the error bound condition (11), we have

$$\begin{aligned} \mathrm{dist}(x_N, S) &\leq c_1 r(x_N)^{c_2} \\ &= c_1 r(x_N)^{c_2} - c_1 r_N(x_N)^{c_2} \\ &= c_1(r(x_N) - r_N(x_N))^{c_2}, \end{aligned}$$

where the equalities follow from the fact that, $r_N(x_N) = 0$ as x_N is the solution to the SAA-SVIP (8). ■

Theorem 3.1 proves the left part of Diagram (9), providing an upper bound of the distance between SAA solutions and the true solutions through the divergence of the SAA residual function $r_N(\cdot)$ and the true one $r(\cdot)$. It paves the way to study the statistical properties of SAA solutions through analysing the inference of SAA residual functions. The validity of theorem 3.1 does not depend on a common error bound constant shared by the sample approximation SVIP (8). So far, it remains unclear how to take Theorem 3.1 as the workhorse for building confidence regions of the true solutions. Without specified expression of the residual function whose structures may be mathematically sophisticated, we can hardly expect desired statistical properties. Fortunately, it turns out to be clear when we specify the residual function $r(\cdot)$. By doing so, the divergence of $r_N(\cdot)$ and $r(\cdot)$ can be characterized in terms of the divergence of SAA function $f_N(\cdot)$ and true one $f(\cdot)$, see e.g. the following Examples 3.1–3.3 for illustration.

Example 3.1 (Normal map error bound condition): Consider SVIP (3) and its SAA counterpart (8). Suppose that the residual function $r(x)$ is given in (12). Then the corresponding SAA residual function corresponding to $r(x)$ is

$$r_N(x) := \|f_N(\Pi_C(x)) + x - \Pi_C(x)\|.$$

Obviously, the normal map error bound condition satisfies the uniform convergence properties in Theorem 3.1. Therefore, there exists a positive constant c such that

$$\begin{aligned} \text{dist}(x_N, S) &\leq c\|f(\Pi_C(x_N)) + x_N - \Pi_C(x_N)\| \\ &= c\|f(\Pi_C(x_N)) + x_N - \Pi_C(x_N) - f_N(\Pi_C(x_N)) + x_N - \Pi_C(x_N)\| \\ &= c\|f(x_N) - f_N(x_N)\|, \end{aligned} \tag{18}$$

where S denotes the set of solutions to SVIP (3). (18) provides us with an upper bound of the distance between the solutions through the divergence of the SAA function $f_N(\cdot)$ and the true counterpart $f(\cdot)$. We are then able to invoke the CLT or ELT for studying the inference of the difference between solutions through the statistical properties of $f_N(x_N) - f(x_N)$. □

Example 3.2 (Natural type error bound condition): Consider the stochastic complementary problem (SCP)

$$0 \leq \mathbb{E}[G(x,\xi)] \perp \mathbb{E}[H(x,\xi)] \geq 0, \tag{19}$$

and denote S as the set of solutions. The SAA counterpart of (19) is:

$$0 \leq G_N(x) \perp H_N(x) \geq 0, \tag{20}$$

where

$$G_N(x) := \frac{1}{N}\sum_{j=1}^{N} G(x,\xi_j) \quad H_N(x) := \frac{1}{N}\sum_{j=1}^{N} H(x,\xi_j)$$

and $\xi_1, \ldots, \xi_N$ is iid sample of ξ. Assume that the natural type error bound condition (14) holds for SCP (19) and x_N is a solution to complementary problem (20). Since the nature type error bound condition satisfies the conditions of

Theorem 3.1, there exists a positive constant c such that

$$\begin{aligned}\operatorname{dist}(x_N, S) &\le c\,\|\min\{\mathbb{E}_P[G(x_N,\xi)], \mathbb{E}_P[H(x_N,\xi)]\}\| \\ &= c\,\|\min\{\mathbb{E}_P[G(x_N,\xi)], \mathbb{E}_P[H(x_N,\xi)]\} - \min\{G_N(x_N), H_N(x_N)\}\| \\ &\le c\|\mathbb{E}_P[G(x_N,\xi)] - G_N(x_N)\| + c\|\mathbb{E}_P[H(x_N,\xi)] - H_N(x_N)\|,\end{aligned}$$

where the second inequality follows from the fact that,

$$|\min\{a_2, b_2\} - \min\{a_1, b_1\}| \le |a_2 - a_1| + |b_2 - b_1|, \quad \forall\, a_1, b_1, a_2, b_2 \in \mathbb{R}.$$

Thereby we are able to provide an upper bound of the distance between the solutions to SAA problem (20) and the true problem (19) through the divergent $\mathbb{E}_P[G(\cdot,\xi)] - G_N(\cdot)$ and $\mathbb{E}_P[H(\cdot,\xi)] - H_N(\cdot)$. □

Example 3.3 (S-type error bound condition): Consider again the stochastic complementary problem (19) and its SAA reformulation (20). Assume that the S-type error bound condition (15) holds and x_N is the solution to SAA problem (20). Similar to Example 3.2, the conditions of Theorem 3.1 hold and therefore a positive constant c exists such that

$$\begin{aligned}&\operatorname{dist}(x_N, S) \\ &= c\,\|(-\mathbb{E}_P[G(x_N,\xi)], -\mathbb{E}_P[H(x_N,\xi)], \mathbb{E}_P[G(x_N,\xi)] \circ \mathbb{E}_P[H(x_N,\xi)])_+\| \\ &= c\Big\|(-\mathbb{E}_P[G(x_N,\xi)], -\mathbb{E}_P[H(x_N,\xi)], \mathbb{E}_P[G(x_N,\xi)] \circ \mathbb{E}_P[H(x_N,\xi)])_+ \\ &\quad - (-G_N(x_N), -H_N(x_N), G_N(x_N) \circ H_N(x_N))_+\Big\| \\ &\le c\Bigg(\Big\|\mathbb{E}_P[G(x_N,\xi)] - G_N(x_N)\Big\| + \Big\|\mathbb{E}_P[H(x_N,\xi)] - H_N(x_N)\Big\| \\ &\quad + \Big\|\mathbb{E}_P[G(x_N,\xi)] \circ \mathbb{E}_P[H(x_N,\xi)] - G_N(x_N) \circ H_N(x_N)\Big\|\Bigg).\end{aligned}$$

Again, we may estimate the distance from SAA solutions to the true solution set through the divergence of functions. Naturally, it is easy to conduct similar analysis with the other type of S-type error bound condition (17). □

3.1. Asymptotic confidence regions: central limit theorem

In this subsection, we shall show the second part of Diagram (9) based on the CLT. As is well known, the CLT lies at the heart of probability theory as it implies that probabilistic and statistical methods that work for normal distributions are also applicable for certain problems involving other types of distributions. For completeness, we recall the classical CLT first. Let $Z \in \mathbb{R}^m$ be a random variable with distribution P_0 and $z_1, z_2, \ldots$ be iid sample of Z. Suppose that Z has covariance Σ. Then

$$\text{(CLT)} \qquad \lim_{N\to\infty} \sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{N} z_j - \mathbb{E}[Z]\right) \Rightarrow N(0,\Sigma),$$

where $\Rightarrow$ means convergence in distribution. We are now ready to present the error bound approach for building the confidence region of solutions to SVIP (3).

Theorem 3.2: *Suppose that* (a) $\{x_N\}$ *is a sequence of solutions to SAA-SVIP* (8) *and* x^* *is a limit point of* $\{x_N\}$, (b) *the normal map error bound condition holds with modulus c,* (c) *the covariance* Σ *of* $F(x^*,\xi)$ *is positive definite. Denote*

$$C_\Sigma^\alpha := \left\{y : y'\Sigma^{-1}y \le \chi^2_{1-\alpha}(n)\right\}$$

and

$$f'_{N'}(\cdot) = \frac{1}{N'}\sum_{j=1}^{N'} F(\cdot,\xi'_j), \tag{21}$$

where $\chi^2_{1-\alpha}(n)$ *is* $1-\alpha$ *quantile of* $\chi^2(n)$ *and* $\xi'_1,\ldots,\xi'_{N'}$ *is independent identically distributed sample of* ξ, *which is also independent with* $\xi_1,\ldots,\xi_N$. *Then*

$$\left\{x : \|x - x_N\| \le c\left(\frac{1}{\sqrt{N'}}\sup_{z\in C_\Sigma^\alpha}\|z\| + \left\|f_N(x_N) - f'_{N'}(x_N)\right\|\right)\right\} \tag{22}$$

defines an approximate $(1-\alpha)$ *confidence region for solutions to SVIP* (3), *that is, it contains a true solution to SVIP* (3) *with probability* $1-\alpha$ *as* $N' \to \infty$.

Proof: Upon combining Theorem 3.1 and the classical CLT, the proof follows directly. In particular, by conditions (a)-(b), Theorem 3.1 holds and hence that

$$\begin{aligned}\mathrm{dist}(x_N,S) &\le c\|f(x_N) - f_N(x_N)\| \\ &= c\left(\left\|f_N(x_N) - f'_{N'}(x_N)\right\| + \left\|f'_{N'}(x_N) - f(x_N)\right\|\right),\end{aligned}$$

where S denotes the set of solutions to SVIP (3), $f(\cdot)$, $f_N(\cdot)$ and $f'_{N'}(\cdot)$ are defined as in (4), (6) and (21) respectively. For any realization of sample $\xi_1,\ldots,\xi_N$ and $\xi'_1,\ldots,\xi'_{N'}$, $\|f_N(x_N) - f'_{N'}(x_N)\|$ is a fixed scalar. Moreover, both $f_N(x_N)$ and $f'_{N'}(x_N)$ converge to $f(x^*)$ with $N, N' \to \infty$ as $x_N \to x^*$ and Lipschitz continuity of $F(\cdot,\xi)$. Then $\|f_N(x_N) - f'_{N'}(x_N)\|$ tends to zero as $N, N' \to \infty$. On the other hand, the CLT implies

$$\sqrt{N'}(f'_{N'}(x_N) - f(x_N)) \Rightarrow N(0,\Sigma).$$

Consequently, we arrive at (22) as C_Σ^α is $1-\alpha$ confidence region to the mean of $N(0,\Sigma)$. ■

Theorem 3.2 establishes an approximate confidence region of the solutions to SVIP (3). Theorem 3.2 relies only on the error bound condition and the classical CLT. Different from the normal map approach, the error bound approach stays valid regardless of the uniqueness or differentiability of the solution mapping with respect to the perturbation of the involved functions. It is worth mentioning that the maximum $\sup_{z\in C_\Sigma^\alpha} \|z\|$ in (22) is not a convex optimization problem. Fortunately, it is somehow easy to calculate its optimal value as C_Σ^α is an ellipsoid. The results presented in Theorem 3.2 can be similarly extended to cases where the normal map error bound condition is replaced by the nature type or S-type error bound conditions. In fact, all the results throughout the rest of this paper hold for the aforementioned three different types of error bound conditions. For simplicity of presentation, our analysis focuses on the situation where normal map error bound condition is under investigation. The extension to other circumstances are puly technical and hence omitted. Moreover, as the covariance matrix Σ in condition (c) of Theorem 3.2 is usually unknown, a natural approximation is the sample covariance matrix of $\{F(x_N, \xi_j')\}_{j=1}^{N'}$; see [12–17] for similar discussions.

Recently, Lamm et al. [13] and Lam and Lu [12] study the component-wise confidence region for the true solution to SVIP (3). As pointed out by [12,13], individual confidence regions of the true solution induce a measure of the uncertainty in each individual component of an SAA solution. Then we are able to assess the uncertainty in an individual component, which thereby allows us to focus on parameters of specific component of our interest. Unfortunately, Theorem 3.2 is not an appropriate tool to construct the component-wise confidence region of the true solutions. This drawback is due to the fact that the error bound condition estimates the distance from a given point to the set of solutions in terms of a unified radius for all components. In fact, if the distance can be represented separately with respect to each single component of variable x, Theorem 3.2 can be used to study component-wise confidence regions. For this purpose, we concentrate on functions with special structures in SVIP (3).

Corollary 3.1: *Suppose that* (a) $\{x_N\}$ *is a sequence of solutions to SAA-SVIP* (8) *and* x^* *is a limit point of* $\{x_N\}$. (b) *The normal map error bound condition holds.* (c) *Function* $F(x, \xi)$ *has separable structures, that is,*

$$F(x, \xi) := F_1(x_1, \xi) + \cdots + F_n(x_n, \xi),$$

where x_i, $1 \le i \le n$, *denotes the ith component of* x. (d) *Fix i and for any given* x_i, *there exists* $\bar{x}_{-i} := (\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n)$ *such that*

$$(f_j)_N(\bar{x}_j) - f_j(\bar{x}_j) = 0, \quad 1 \le j \ne i \le n,$$

where

$$(f_j)_N(x_j) := \frac{1}{N}\sum_{k=1}^{N} F_j(x_j,\xi_k) \quad \text{and} \quad f_j(x_j) := \mathbb{E}_P[F_j(x_j,\xi)], \quad j=1,\dots,n. \tag{23}$$

(e) *The covariance* Σ_i *of* $F_i((x^*)_i,\xi)$, $1 \le i \le n$ *is positive definite, where* $(x^*)_i$ *is the limit of sequence* $\{(x_N)_i\}$. *Define*

$$C^{\alpha}_{\Sigma_i} := \left\{y : y'\Sigma_i^{-1}y \le \chi^2_{1-\alpha}(n)\right\}$$

and

$$(f'_i)_{N'}(\cdot) = \frac{1}{N'}\sum_{j=1}^{N'} F_i(\cdot,\xi'_j), \tag{24}$$

where $\chi^2_{1-\alpha}(n)$ *is* $1-\alpha$ *quantile of* $\chi^2(n)$ *and* $\xi'_1,\dots,\xi'_{N'}$ *is independent identically distributed sample of* ξ, *which is also independent with* $\xi_1,\dots,\xi_N$. *Then the set*

$$\left\{x_i : |x_i-(x_N)_i| \le c\left(\frac{1}{\sqrt{N'}}\sup_{z\in C^{\alpha}_{\Sigma_i}}\|z\| + \left\|(f_i)_N((x_N)_i)-(f'_i)_{N'}((x_N)_i)\right\|\right)\right\}$$

defines an approximate $(1-\alpha)$ *confidence region for ith component of solutions to SVIP* (3).

Proof: Denote x_i as the ith component of x and $S_i := \{x_i : x\in S\}$. By normal error bound conditions, Example 3.1 implies that

$$\begin{aligned}
&\text{dist}(x_i,S_i)\\
&\quad\le \text{dist}((x_i,\bar{x}_{-i}),S)\\
&\quad= c\|(f_1)_N(\bar{x}_1)+\cdots+(f_i)_N(x_i)+\cdots+(f_n)_N(\bar{x}_n)-f_1(\bar{x}_1)\\
&\qquad -\cdots-f_i(x_i)\cdots-f_n(\bar{x}_n))\|\\
&\quad= c\|(f_i)_N(x_i)-f_i(x_i)\|\\
&\quad\le c(\|(f_i)_N(x_i)-(f'_i)_{N'}(x_i)\|+\|(f'_i)_{N'}(x_i)-f_i(x_i)\|),
\end{aligned}$$

where the equalities follow from conditions (c) and (d). The rest of this proof follows similarly from the discussion in Theorem 3.2. ■

Corollary 3.1 studies the component confidence regions of the solutions to SVIP (3). Conditions (c) and (d) in Corollary 3.1 ensure the component error bound conditions hold and then the proof is to mimic the proof of Theorem 3.2. A prevailing example which meets the assumptions in Corollary 3.1 is the stochastic linear variational inequalities: find x such that

$$0 \leq x \perp \mathbb{E}_P[A(\xi)x + q(\xi)] \geq 0.$$

In particular, conditions (c) holds with

$$F(x, \xi) := A_1(\xi)x_1 + \cdots + (A_i(\xi)x_i + q(\xi)) + \cdots + An(\xi)x_n,$$

where $A_i(\xi)$ denotes the ith column of matrix $A(\xi)$. Condition (d) holds with $\bar{x}_j = 0, 1 \leq j \neq i \leq n$.

As shown in [16], Assumptions 2.1-2.2 serve as sufficient conditions for the following functional central limit theorem.

Proposition 3.1 (Functional CLT): [16, Theorem 4.3] *Consider the SVIP (3) and its SAA counterpart (8). Let $F(\cdot)$ satisfy Assumption 2.1. Then there exists a $C^1(X, \mathbb{R}^n)$ valued random variable Y such that for each finite subset $\{x^1, x^2, \ldots, x^m\} \subseteq \mathcal{C}$, the random vector*

$$(Y(x^1), \ldots, Y(x^m))$$

has a multivariate normal distribution with zero mean and the same covariance matrix as that of

$$(F(x^1, \xi), \ldots, F(x^m, \xi))$$

and as $N \to \infty$, $\sqrt{N}(f_N(\cdot) - f(\cdot))$ converges in distribution, in $C^1(\mathcal{C}, \mathbb{R}^n)$, to Y.

Upon combining the functional CLT in Proposition 3.1 and the discussion for Example 2.1, we have the following results.

Theorem 3.3: *Suppose that* (a) *Assumptions 2.1-2.2 hold,* (b) *$\{x_N\}$ is a sequence of solutions to SAA-SVIP (8) and x^* is a limit point,* (c) *the covariance Σ of $f(x^*, \xi)$ is positive definite. Denote the $1 - \alpha$ confidence region of normal distribution $N(0, \Sigma)$ as C^α_Σ. Then, there exists a bounded constant c such that the set*

$$\left\{ x : \|x - x_N\| \leq \frac{c}{\sqrt{N}} \sup_{z \in C^\alpha_\Sigma} \|z\| \right\}$$

defines an approximate $(1 - \alpha)$ confidence region for solutions to SVIP (3).

Proof: Example 2.1 shows that the normal map error bound condition holds under the Assumptions 2.1– 2.2. Then the rest follows from Proposition 3.1 and Theorem 3.2 directly. ■

Theorem 3.3 shows that the error bound approach is applicable under Assumptions 2.1– 2.2. It seems that the normal map approach can also be regarded as local error bound approach with more exactness. The normal map approach requires the directional derivative of the solution map. The norm of such directional derivative may act as the local error bound constant. Obviously, the error bound constant, especially the global error bound constant mentioned in Examples 3.1–3.3 may be greater than the norm of directional derivative as little information of the limit point x^* is required. It seems to us that the confidence region built through error bound approach may be bigger than the one built through normal map approach. Without the exactness of confidence regions, can we still claim that the error bound approach is beneficial? The answer is affirmative! In fact, the first benefit relies on the fact that error bound approach does not require the uniqueness of the solution to SVIP (3). Moreover, it permits us to construct the asymptotic confidence regions through empirical likelihood approach or even to build the non-asymptotic confidence regions. We present the details in the next two subsections.

3.2. Asymptotic confidence regions: empirical likelihood method based

Both CLT and Delta method have been widely used for studying the properties of statistical estimates of the stochastic programming when sample average approximation is considered. In comparison, the empirical likelihood method received less attention in the stochastic optimization literature. Recently, empirical likelihood approach has been used to study the inference of the stochastic programming problem; for example, Lam and Zhou [20] construct the confidence region for the optimal value; Lam [21] and Duchi et al. [22] study statistical inference and distributionally robust solution methods for stochastic optimization problems. Motivated by [20–22], in order to build the confidence region for the true solution of SVIP (3), we penetrate the error bound condition into the empirical likelihood method.

The basic theory of empirical likelihood method is Owen's empirical likelihood theorem (ELT) [18]. Different from CLT and Delta method, ELT builds confidence regions through solving distributionally robust optimization problems with ambiguity set which is defined as a divergence-based ball. In the rest of this section, we mainly focus on the Kullback-Leibler (KL) divergence, while our results can be directly extended to other divergences.

The KL divergence originates from the field of information theory. Interestingly, it gains popularity recently in distributionally robust optimization; see, e.g. [21,23,24] for some recent developments. For the case where ξ is a discrete random variable, the KL divergence is defined as

$$d_{\mathrm{KL}}(Q,P) = \sum_i P(i) \ln \frac{P(i)}{Q(i)};$$

on the other hand, if ξ is a continuous random variable,

$$d_{\mathrm{KL}}(Q,P) = \int_{-\infty}^{\infty} p(\xi) \ln \frac{p(\xi)}{q(\xi)}\, \mathrm{d}\xi,$$

where $P(i)$, $Q(i)$ denote distributions and $p(\xi)$, $q(\xi)$ are associated density functions.

Proposition 3.2 ([22, Proposition 1]): *Let $z_1, \ldots, z_N$ be independent and identically distributed (iid) sample of random vector Z. Suppose that the distribution P_0 of Z has finite covariance with rank $d_0 \le d$. Then*

$$\lim_{N\to\infty} \text{Prob}\left\{\mathbb{E}_{P_0}[Z] \in \left(\mathbb{E}_P[Z] : d_{\text{KL}}(P, P_N) \le \frac{\rho}{N}\right)\right\} = \text{Prob}\left\{\chi^2(d_0) \le \rho\right\}. \tag{25}$$

According to Proposition 3.2, the probability of the true mean contained in the region which is constructed through KL divergence converges to χ^2-distribution with freedom d_0. This nice property has been widely used in distributionally robust optimization [22].

Upon combining Proposition 3.2 and Theorem 3.1, we may build the confidence region of the solutions to SVIP (3) through the empirical likelihood method.

Theorem 3.4: *Assume that the normal map error bound condition holds. Denote*

$$\begin{aligned} d'_{x_N} := \quad & \max_P \quad \|\mathbb{E}_P[F(x_N, \xi)] - f'_{N'}(x_N)\| \\ & \text{s.t.} \quad d_{\text{KL}}(P, P_{N'}) \le \frac{\rho}{N'}, \end{aligned} \tag{26}$$

where $f'_{N'}(\cdot)$ is defined in (21) and $P_{N'}$ denotes empirical distributions

$$P_{N'} := \frac{1}{N'}\sum_{k=1}^{N'} \mathbb{I}_{\xi'_k}(\omega), \quad \text{and} \quad \mathbb{I}_{\xi'_k}(\omega) := \begin{cases} 1, & \text{if } \xi = \xi'_k, \\ 0, & \text{if } \xi \ne \xi'_k. \end{cases}$$

Suppose further that $F(x_N, \xi)$ has finite covariance with rank $\gamma_N \le n$ and ρ is chosen such that $P(\chi^2(\gamma_N) \le \rho) = 1 - \alpha$. Then

$$\left\{x : \|x - x_N\| \le c\left(d'_{x_N} + \left\|f_N(x_N) - f'_{N'}(x_N)\right\|\right)\right\}$$

defines an approximate $(1 - \alpha)$ confidence region for solutions to SVIP (3).

Proof: As the normal map error bound condition holds, Example 3.1 states that

$$\begin{aligned} \text{dist}(x_N, S) &\le c\|(f(x_N) - f_N(x_N))\| \\ &= c\left\|f_N(x_N) - f'_{N'}(x_N)\right\| + \left\|f'_{N'}(x_N) - f(x_N)\right\|, \end{aligned}$$

where S denotes the set of solutions to SVIP (3). By Proposition 3.2,

$$\lim_{N'\to\infty} \text{Prob}\left\{\left\|f'_{N'}(x_N) - f(x_N)\right\| \le d'_{x_N}\right\} = \text{Prob}\left\{\chi^2(\gamma_N) \le \rho\right\}.$$

We may arrive at the results by combining the two formulations above directly. ∎

It is usually a difficult task to estimate the covariance matrix of $F(x^*, \xi)$. On the other hand, it should be easier to estimate the rank of that matrix. As the tradeoff, we need to calculate the constant d'_{x_N} through solving an optimization problem (26). Note that in problem (26), the variable is distribution P, which must have the same support set with empirical distribution $P_{N'}$. Note further that $P_{N'}$ is induced by the iid sample $\xi'_1, \dots, \xi'_N$, consequently (26) is an optimization problem in a finite dimensional Euclidean space. Unfortunately, problem (26) is not a convex optimization problem as the objective is to maximize a convex function. Even though we understand that the maximum value of a convex function is taken at the boundary point, it is still difficult to calculate the exact value d'_{x_N}. One possible way to overcome the non-convexity is to consider a component-relaxed confidence interval. In particular, we shall denote

$$(\lambda'_{x_N})_i := \begin{array}{ll} \max\limits_{P} & \left| \left(\mathbb{E}_P[F(x_N, \xi)] \right)_i - \left(f'_{N'}(x_N) \right)_i \right| \\ \text{s.t.} & d_{\text{KL}}(P, P_{N'}) \le \dfrac{\rho}{N'}, \end{array} \tag{27}$$

and define the approximate $(1-\alpha)$ confidence region as:

$$\left\{ x : \|x_N - x\| \le c \left(\|\lambda'_{x_N}\| + \|f'_{N'}(x_N) - f(x_N)\| \right) \right\},$$

where $\lambda'_{x_N} = ((\lambda'_{x_N})_1, \dots, (\lambda'_{x_N})_n)^T$. Obviously, $d'_{x_N} \le \|\lambda'_{x_N}\|$ as maximizing

$$\left| \left(\mathbb{E}_P[F(x_N, \xi)] \right)_i - \left(f'_{N'}(x_N) \right)_i \right|, \quad i = 1, \dots, n$$

releases the dependence between the components of vector $\mathbb{E}_P[F(x_N, \xi)] - f'_{N'}(x_N)$. Furthermore, for $i = 1, \dots, n$, denote

$$\overline{(\lambda'_{x_N})_i} := \begin{array}{ll} \max\limits_{P} & \left(\mathbb{E}_P[F(x_N, \xi)] \right)_i \\ \text{s.t.} & d_{\text{KL}}(P, P_{N'}) \le \dfrac{\rho}{N'}, \end{array} \tag{28}$$

and

$$\underline{(\lambda'_{x_N})_i} := \begin{array}{ll} \min\limits_{P} & \left(\mathbb{E}_P[F(x_N, \xi)] \right)_i \\ \text{s.t.} & d_{\text{KL}}(P, P_{N'}) \le \dfrac{\rho}{N'}. \end{array} \tag{29}$$

Then

$$(\lambda'_{x_N})_i = \max \left\{ \overline{(\lambda'_{x_N})_i} - (f'_{N'}(x_N))_i, (f'_{N'}(x_N))_i - \underline{(\lambda'_{x_N})_i} \right\}, \quad i = 1, \dots, n.$$

Fortunately, (28) and (29) are convex optimization problems which can be solved efficiently by a wide range of commercial optimization solvers (such as CVX developed by Grant and Boyd [25]).

The empirical likelihood method can be used to construct the componentwise confidence intervals if function $F(x, \xi)$ has separable structure.

Corollary 3.2: *Suppose that* (a) *the normal map error bound condition holds.* (b)

$$F(x, \xi) := F_1(x_1, \xi) + \cdots + F_n(x_n, \xi).$$

(c) *Fix i and for any given x_i, there exists $\bar{x}_{-i} := (\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n)$ such that*

$$(f_j)_N(\bar{x}_j) - f_j(\bar{x}_j) = 0, \quad 1 \le j \ne i \le n,$$

where $(f_j)_N(\cdot)$ and $f_j(\cdot)$ are defined as in (23). (d) *$F_i((x_N)_i, \xi)$ has finite covariance with rank $\gamma_N \le n$. Denote*

$$\begin{aligned} d'_{(x_N)_i} := \quad & \max_P \quad \|\mathbb{E}_P[F_i((x_N)_i, \xi)] - (f'_i)_{N'}((x_N)_i)\| \\ & \text{s.t.} \quad d_{\mathrm{KL}}(P, P_{N'}) \le \frac{\rho}{N'}, \end{aligned}$$

and

$$P(\chi^2(\gamma_N) \le \rho) = 1 - \alpha,$$

where $(f'_i)_{N'}(\cdot)$ is defined as in (24). *Then interval*

$$\left\{ x_i : \|x_i - (x_N)_i\| \le c \left(d'_{(x_N)_i} + \left\| (f_i)_N((x_N)_i) - (f'_i)_{N'}((x_N)_i) \right\| \right) \right\}$$

defines an approximate $(1 - \alpha)$ *confidence region for ith component of solutions to SVIP* (3).

Through a self-normalization property, the empirical likelihood method works without any knowledge or estimation of unknown quantities, such as covariance matrix. It is worth mentioning that, the relaxation technique for tackling non-convexity may cause unfavourable inexactness of the confidence regions, especially when the dimension of variable x is large.

3.3. Non-asymptotic confidence regions

Sections 3.1 and 3.2 construct the asymptotic confidence region of the solutions to SVIP (3) based on asymptotic distributions. When the sample size is large, asymptotic distributions present a precise characterization of the confidence region. However, asymptotic confidence regions may provide limited information about the quality of the estimate when the sample size is small. One of our contributions is to fill in this gap by studying the non-asymptotic (uniformly for any sample size N) confidence regions of the solutions to SVIP (3), which suffice to meet the needs of practical interest. Indeed, in order to get practically useful confidence regions, it is not necessary that the sequence possesses an asymptotic distribution [26]. The boundedness in probability uniformly of random variables is sufficient enough. Namely, a sequence of random variables is bounded in probability with normal tails if for any $\epsilon > 0$, there exist constants β_1 and β_2 such that

$$\sup_N \text{Prob}\left\{|Z_N| \geq \epsilon\right\} \leq \beta_1 e^{-\beta_2 \epsilon^2}.$$

The boundedness in probability uniformly helps to construct the non-asymptotic confidence regions of optimal value of optimization problems, see e.g. [26]. We consider to use this property for constructing the non-asymptotic confidence regions of the solutions of SVIP (3) in this part.

The following lemma characterizes the bounds on probability of large deviations of martingales, which plays a key role in our analysis.

Lemma 3.1 ([27, Lemma A.1]): *Let $d_1, d_2, \ldots$ be a scalar martingale-difference such that for some $\sigma > 0$ it holds*

$$\mathbb{E}[e^{d_t^2/\sigma^2} | \xi_1, \ldots, \xi_{t-1}] \leq e \quad \text{a.s. } t = 1, 2, \ldots.$$

Then

$$\text{Prob}\left\{\sum_{t=1}^{N} d_t > \lambda\sigma\sqrt{N}\right\} \leq \begin{cases} e^{\frac{-\lambda^2}{4\tau^*}}, & 0 \leq \lambda \leq 2\sqrt{\tau^* N}, \\ e^{\frac{-\lambda^2}{3}}, & \lambda > 2\sqrt{\tau^* N}, \end{cases}$$

where $\tau^* = 0.557409\cdots$. is the smallest positive real such that

$$e^t \leq t + e^{\tau^* t^2}, \quad \forall\, t \in \mathbb{R}.$$

Thanks to Lemma 3.1, the sum of a martingale-difference sequence is bounded in probability uniformly with normal tails. Obviously, we observe that for iid samples $\xi_1, \xi_2, \ldots$ and any given $x \in \mathcal{C}$, the sequence

$$F(x, \xi_1) - \mathbb{E}_P[F(x, \xi_1)], F(x, \xi_2) - \mathbb{E}_P[F(x, \xi_2)], \ldots$$

is a martingale-difference sequence. This observation thereby allows us to call Lemma 3.1 together with global error bound conditions for constructing the non-asymptotic confidence regions of the solutions to SVIP (3).

Theorem 3.5: *Suppose that*

(a) *x_N is a sequence of solutions to SAA-SVIP (8).*
(b) *The global normal map error bound condition holds with modulus c.*
(c) *Denote*

$$(d_i)_j := \left(F(x_N, \xi'_j) - \mathbb{E}_P[F(x_N, \xi)]\right)_i, \quad j = 1, \ldots, N',$$

where $\xi'_1, \ldots, \xi'_{N'}$ is independent identically distributed sample of ξ, which is also independent with $\xi_1, \ldots, \xi_N$. For $i = 1, \ldots, n$, there exists a positive *constant σ_i such that*

$$\mathbb{E}[e^{(d_i)_j/\sigma_i^2}] \le e, \quad j = 1, 2, \dots, N'.$$

Then, the set

$$\left\{ x : \|x - x_N\| \le c \left(\frac{\lambda\sqrt{\sigma_1^2 + \cdots + \sigma_n^2}}{\sqrt{N'}} + \left\| f_N(x_N) - f'_{N'}(x_N) \right\| \right) \right\} \tag{30}$$

defines an $(1 - 2e^{\frac{-\lambda^2}{4\tau^*}})^n$ confidence region for solutions to SVIP (3), where n is the dimension of x.

Proof: Thanks to condition (a), Theorem 3.1 is valid, which results in that

$$\begin{aligned} \mathrm{dist}(x_N, S) &\le c\|(f(x_N) - f_N(x_N))\| \\ &= c \left\| f_N(x_N) - f'_{N'}(x_N) \right\| + \left\| f'_{N'}(x_N) - f(x_N) \right\|, \end{aligned}$$

where S denotes the set of solutions to SVIP (3). Now we are in the position to invoke Lemma 3.1 in order to provide an upper bound for $\|f(x_N) - f'_{N'}(x_N)\|$ in probability. Since $\xi'_1, \xi'_2, \dots, \xi'_{N'}$ is iid sample, for each $i = 1, \dots, n$, $(d_i)_1, (d_i)_2, \dots, (d_i)_{N'}$ is a scalar martingale-difference sequence. Thanks to condition (c), Lemma 3.1 is applicable. As a consequence,

$$\mathrm{Prob}\left\{ \left| \sum_{t=1}^{N} (d_i)_j \right| > \lambda\sigma\sqrt{N'} \right\} \le 2e^{\frac{-\lambda^2}{4\tau^*}}.$$

By easy calculation, we may then arrive at the conclusion. ∎

Theorem 3.5 actually provides a universal confidence regions of the solutions to the SVIP (3). In particular, with desired probability level and constant λ chosen personally, (30) provides a confidence region for each given sample size N and N'. As one may observe, the confidence region relie parameters λ and sample size N and N'. For a given sample size, we may increase λ in order to provide an $(1 - \epsilon)$-confidence region, where ϵ is small positive number. On the other hand, for a confidence region with fixed radius, the decision maker may decide to stop or to increase the sample size by continuing sampling and data collection. We should also note that the confidence regions of Theorem 3.5 involve some positive constants $\sigma_1, \dots, \sigma_n$, defined in (4). The valid upper bounds on these constants are crucial to obtain the confidence regions. To the best of our knowledge, there is no generic procedure which allows us to construct such estimates. If for $i = 1, \dots, n$, the values of $(F(x_N, \xi))_i$ fall into a bounded interval $[-\sigma, \sigma]$, condition (c) of Theorem 3.5 holds. This assumption can be satisfied when $(F(x_N, \xi))_i$ is continuous in ξ and the support set of ξ is bounded. For unbounded cases, we refer to [26] for more discussions.

Similarly, if function $F(x,\xi)$ has separable structure, we may construct component-wise non-asymptotic confidence regions of the solutions to SVIP (3).

Corollary 3.3: *Suppose that* (a) $\{x_N\}$ *is a sequence of solutions to SAA-SVIP* (8). (b) $F(x,\xi) := F_1(x_1,\xi) + \cdots + F_n(x_n,\xi)$. (c) *Fix* i *and for any given* x_i*, there exists* $\bar{x}_{-i} := (\bar{x}_1,\ldots,\bar{x}_{i-1},\bar{x}_{i+1},\ldots,\bar{x}_n)$ *such that*

$$(f_j)_N(\bar{x}_j) - f_j(\bar{x}_j) = 0, \quad 1 \le j \ne i \le n,$$

where $(f_j)_N(\cdot)$ *and* $f_j(\cdot)$ *are defined as in* (23). (d) *The normal map error bound condition holds.* (e) *For* $j = 1,\ldots,n$*, there exists a positive constant* σ_j *such that*

$$\mathbb{E}[e^{(F_i((x_N)_i,\xi'_\kappa))_j - (\mathbb{E}[F_i((x_N)_i,\xi)])_j/\sigma_j^2}] \le e, \quad \kappa = 1,2,\ldots,N'.$$

Then, the interval

$$\left\{x_i : \|x_i - (x_N)_i\| \le c\left(\frac{\lambda\sqrt{\sigma_1^2+\cdots+\sigma_n^2}}{\sqrt{N'}} + \left\|(f_i)_N((x_N)_i) - (f'_i)_{N'}((x_N)_i)\right\|\right)\right\} \tag{31}$$

defines an $(1-2e^{\frac{-\lambda^2}{4\tau^*}})^n$ *confidence region for ith component of solutions to SVIP* (3)*, where* $(f'_i)_{N'}(\cdot)$ *is defined as in* (24).

4. Numerical results

In this section, we report some preliminary numerical results on a stochastic linear complementarity problem which has been studied in [15,16].

Example 4.1 ([16, Example in Section 6.2]): Consider a stochastic linear complementarity problem:

$$0 \le \mathbb{E}[F(x,\xi)] \perp x \ge 0, \tag{32}$$

where

$$F(x,\xi) = \begin{bmatrix}\xi_1 & \xi_2\\ \xi_3 & \xi_4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\xi_5\\ \xi_6\end{bmatrix}$$

and ξ follows uniform distribution over the box:

$$\left\{\xi \in \mathbb{R}^6 | (0,0,0,0,-1,-1) \le \xi \le (2,1,2,4,1,1)\right\}.$$

Then,

$$f(x) = M_0 x + b_0 \quad \text{with } M_0 := \begin{bmatrix}1 & 1/2\\ 1 & 2\end{bmatrix}, \quad b_0 = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

Obviously, M_0 is a positive definite matrix (without the symmetry), and then it is a P-matrix (see Example 2.2 for the definition). Note that the minimum singular value of M_0 is 0.6193, and the maximum is 2.4221. The constant γ in the definition of P-function is 0.6193/2 and the Lipschitz constant of $f(x)$ is 2.4221. Following the analysis of Example 2.2, the global natural type error bound condition holds with constant $c = \frac{1+L}{\gamma} = 11.0515$, that is,

$$\mathrm{dist}(x_N, S) \leq \; 11.0515\|f(x_N) - f_N(x_N)\|,$$

where S denotes the set of solutions to (32),

$$f_N(x) := M_N x + b_N,$$

M_N and b_N are the SAA counterparts of M_0 and b_0 respectively. Then, CLT based Theorem 3.2 and ELT based Theorem 3.4 can be used to build the confidence region of the true solutions to (32).

Moreover, it is easy to verify that for $i = 1, 2$

$$\mathrm{dist}((x_N)_i, S_i) \leq \; 11.0515\|f_i((x_N)_i) - (f_i)_N((x_N)_i)\|,$$

where $S_i := \{x_i : x \in S\}$,

$$f_i(x_i) := (M)_i x_i + b_0 \quad \text{and} \quad (f_i)_N(x_i) := (M_N)_i x_i + b_N,$$

and $(A)_i$ denotes the ith column of matrix A. Then Corollaries 3.1 and 3.2 can be used to construct the component-wise confidence region of solutions to (32).

Note also that the support set of Ξ is a compact set, for any given x and $i = 1$, 2, there exist bounded constants

$$\sigma_1 := \sup_{\xi \in \Xi} (F(x, \xi))_1 = 2|x_1| + |x_2| + 1 \quad \text{and}$$

$$\sigma_2 := \sup_{\xi \in \Xi} (F(x, \xi))_2 = 2|x_1| + 4|x_2| + 1$$

satisfy the condition (c) of Theorem 3.5. Since the P function implies the global error bound condition holds (Example 2.2), we may use the results in section 3.3 to construct the non-asymptotic confidence regions of the solutions to (32).

Based on the analysis above, we do the numerical test with samples size 10, 100, 500 and 1000. The corresponding sample average matrix M_N and b_N are

$$M_{10} = \begin{bmatrix} 1.1361 & 0.6417 \\ 1.1165 & 2.2844 \end{bmatrix} \quad b_{10} = \begin{bmatrix} -0.1808 \\ -0.4552 \end{bmatrix} \quad x_{10} = \begin{bmatrix} 0.0644 \\ 0.1678 \end{bmatrix}$$
$$M_{100} = \begin{bmatrix} 0.9870 & 0.5333 \\ 0.9812 & 2.0358 \end{bmatrix} \quad b_{100} = \begin{bmatrix} -0.0983 \\ -0.1317 \end{bmatrix} \quad x_{100} = \begin{bmatrix} 0.0873 \\ 0.0226 \end{bmatrix}$$
$$M_{500} = \begin{bmatrix} 0.9704 & 0.5195 \\ 1.0037 & 2.0721 \end{bmatrix} \quad b_{500} = \begin{bmatrix} -0.0129 \\ -0.0474 \end{bmatrix} \quad x_{500} = \begin{bmatrix} 0.0014 \\ 0.0222 \end{bmatrix}$$
$$M_{1000} = \begin{bmatrix} 0.9829 & 0.5029 \\ 0.9704 & 1.9983 \end{bmatrix} \quad b_{1000} = \begin{bmatrix} -0.0049 \\ -0.0062 \end{bmatrix} \quad x_{1000} = \begin{bmatrix} 0.0045 \\ 0.0009 \end{bmatrix}$$

For each given sample, we solve the LCP by the LCP solver developed by Tassa [28], which returns the solutions x_{10}, x_{100}, x_{500} and x_{1000} as shown above. For

Table 1. Asymptotic confidence regions: CLT.

	N			
Prob	10	100	500	1000
90%	0.7965	0.3457	0.0865	0.0538
95%	0.8531	0.3655	0.0945	0.0594
97.5%	0.9035	0.3830	0.1016	0.0643

Table 2. Asymptotic confidence regions: Owen's ELT.

	N			
Prob	10	100	500	1000
90%	1.0764	0.4562	0.1393	0.0917
95%	1.1613	0.4913	0.1547	0.1026
97.5%	1.2310	0.5225	0.1685	0.1123

Table 3. Non-asymptotic confidence regions.

	N			
Prob	10	100	500	1000
90%	1.7180	0.6586	0.2180	0.1428
95%	1.8401	0.7004	0.2353	0.1546
97.5%	1.9515	0.7385	0.2512	0.1654

the convex problem induced by empirical likelihood theorem in Section 3.2, we employ the solver CVX (version 1.22) developed by Grant and Boyd [25] to solve it. Moreover, we set $N = N'$.

Tables 1–3 report the radius γ of the confidence regions:

$$\text{Prob}\,\{\mathcal{B}(x_N, c\gamma) \text{ contains a solution of (32)}\} \geq 1 - \alpha$$

with $\alpha = 10\%, 5\%$ and 2.5% and $c = 11.0515$. We can observe from Tables 1–3 that for the given sample size N, the radius is increasing with the decreasing of probability parameter α, and for the fixed probability parameter α, the radius is decreasing with the increasing of the sample size N. Moreover, for the given sample size N and probability parameter α, the confidence region returned by CLT (Subsection 3.1) is more exact than Owen's ELT (Subsection 3.2) and non-asymptotic confidence regions (Subsection 3.3). The underlying reason may be that (1) the CLT based confidence regions have used more information of the random variable $f(x_N, \xi)$, such as the covariance matrix, but Owen's ELT based confidence regions only use the rank information of the covariance matrix, (2) the componentwise relaxation has been used to handle the non-convexity of the optimization problem involving Owen's ELT method. Naturally, the non-asymptotic confidence regions should be bigger than the asymptotic confidence regions as it holds for any given sample rather than for large sample size.

We also test the component-wise confidence regions of the solutions to SVIP (32). However, as the results reported in Table 4, radius of the componentwise confidence regions is almost same as the confidence region of whole

Table 4. Component's asymptotic confidence regions: CLT.

	N			
Prob	10	100	500	1000
90%	(0.7544, 0.8193)	(0.3517, 0.3385)	(0.0864, 0.0866)	(0.0538, 0.0537)
95%	(0.8044, 0.8772)	(0.3713, 0.3581)	(0.0944, 0.0946)	(0.0593, 0.0593)
97.5%	(0.8489, 0.9287)	(0.3888, 0.3756)	(0.1015, 0.1017)	(0.0643, 0.0643)

Table 5. (Component) Asymptotic confidence regions: CLT.

	N			
Prob	10	100	500	1000
90%	1.8831	0.4604	0.1867	0.1819
	(1.0441, 1.7877)	(0.3023, 0.4566)	(0.1241, 0.1695)	(0.1208, 0.1384)
95%	2.0514	0.5053	0.2059	0.1958
	(1.1338, 1.9038)	(0.3268, 0.4976)	(0.1350, 0.1875)	(0.1286, 0.1509)
97.5%	2.2012	0.5453	0.2229	0.2082
	(1.2138, 2.0717)	(0.3485, 0.5340)	(0.1448, 0.2036)	(0.1355, 0.1621)

solutions. The underlying reason may be that the solutions of the sample approximation of (32) is x_{10}, x_{100}, x_{500} and x_{1000} which are too close to $(0, 0)$ and then the covariance of the row and column of

$$\begin{bmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \xi_5 \\ \xi_6 \end{bmatrix}$$

is almost determined by random variable ξ_5 and ξ_6.

In the next, we make a little change of the problem (32) by adding a constant vector on $F(x, \xi)$, that is, consider

$$0 \le \mathbb{E}[F(x, \xi)] \perp x \ge 0, \tag{33}$$

where

$$F(x, \xi) = \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 3.0 \end{bmatrix} + \begin{bmatrix} \xi_5 \\ \xi_6 \end{bmatrix}$$

and ξ follows uniform distribution over the same box. Obviously, $(0, 0)$ is not a solution to (33) and $(1, 1)$ is. We construct the componentwise confidence regions with same four group sample. Tables 4–7 record the corresponding radius where in each line the number above (a, b) denotes the radius of the confidence regions of the whole solutions to SVIP and a, b denotes the radius of the confidence regions to first component x_1 and second component x_2 respectively. It is easy to obverse that the radius of the component confidence regions have the tendency with the radius of whole confidence regions.

5. Summary

This paper aims to construct the confidence regions of SVIP through an error bound approach. Compared to the established normal map approach

Table 6. (Component) Asymptotic confidence regions: Owen's ELT.

	N			
Prob	10	100	500	1000
90%	2.5944	0.6666	0.2767	0.2460
	(1.4788, 2.4437)	(0.4707, 0.6211)	(0.1963, 0.2451)	(0.1753, 0.1908)
95%	2.8300	0.7423	0.3087	0.2690
	(1.6139, 2.6619)	(0.5204, 0.6859)	(0.2177, 0.2739)	(0.1909,0.2108)
97.5%	3.0204	0.8102	0.3373	0.2895
	(1.7249, 2.8400)	(0.5514, 0.7386)	(0.2367 0.2995)	(0.2047, 0.2286)

Table 7. (Component) Non-asymptotic confidence regions.

	N			
Prob	10	100	500	1000
90%	3.9856	1.1366	0.5814	0.4842
	(2.6660, 3.2800)	(0.8310, 0.9225)	(0.3580, 0.3850)	(0.2915, 0.2841)
95%	4.2897	1.2283	0.6303	0.5212
	(2.8722, 3.5136)	(0.8950, 0.9916)	(0.3865, 0.4163)	(0.3121, 0.3056)
97.5%	4.5675	1.3122	0.6751	0.5550
	(3.0624, 3.7291)	(0.9542, 1.0553)	(0.4128, 0.4452)	(0.3312, 0.3253)

[12–17,29] which analyses the asymptotic distributions of solutions, the error bound approach constructs the confidence regions through empirical likelihood theory or large deviation theory.

As the error bound approach does not rely on Delta method, information of the differential of normal map is not required. This advantage allows us to avoid assuming uniqueness of the solution to SVIP. Moreover, the error bound approach may construct the non-asymptotic confidence regions of the solutions to SVIP in terms of the large deviation theorem. As far as we know, this is the first result on the non-asymptotic confidence regions of the solutions to SVIP, which may shed some light on balancing solutions' accuracy and samples' amount.

Notes

1. The uniqueness and Lipschitz of $z(\cdot)$ is with respect to the perturbation of the function $f(\cdot)$ in SVIP (3).
2. For a given point, the difference of the SAA function and the true one can be taken as a random vector.
3. $df(x_0)(\cdot)$ denotes the B-derivative, see [16, Page 547] for details.
4. $C^1(\mathcal{C}, \mathbb{R}^n)$ denotes the Banach space of continuously differentiable mappings $g : \mathcal{C} \to \mathbb{R}^n$.
5. The ball is defined in the Banach space of continuously differentiable mappings $f : \mathcal{C} \to \mathbb{R}^n$, equipped with the norm defined as in [16, (9) page 4].
6. A solution $x^* \in \mathbb{R}^n$ to complementarity problem (16) is said to be non-degenerate if $x_i^* \neq (h(x^*))_i, i = 1, \ldots, n$.
7. The corresponding residual function $r_N(\cdot)$ means that $r_N(\cdot)$ and $r(\cdot)$ are induced by the same type error bound conditions, such as, normal map error bound condition. But $r_N(\cdot)$ is corresponding to SAA-SVIP (8) and $r(\cdot)$ is corresponding to SVIP (3). Please see Examples 3.1–3.3 for details.

Acknowledgements

We would like to thank the editor for organizing an effective review and two anonymous referees for insightful comments and constructive suggestions which help us significantly to consolidate the paper. The research is supported by the NSFC grants #11971090, #11971220, Fundamental Research Funds for the Central Universities under grant DUT19LK24 and Guangdong Basic and Applied Basic Research Foundation 2019A1515011152.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by National Natural Science Foundation of China 11971090 and 1971220.

References

[1] Facchinei F, Pang J-S. Finite-dimensional variational inequalities and complementarity problems. New York: Springer; 2003.
[2] Chen X, Fukushima M. Expected residual minimization method for stochastic linear complementarity problems. Math Oper Res. 2005;30:1022–1038.
[3] Gürkan G, Özge AY, Robinson SM. Sample-path solution of stochastic variational inequalities. Math Program. 1999;84:313–333.
[4] Gwinner J. A class of random variational inequalities and simple random unilateral boundary value problems: existence, discretization, finite element approximation. Stoch Anal Appl. 2000;18:967–993.
[5] Causa A, Jadamba B, Raciti F. A migration equilibrium model with uncertain data and movement costs. Decisions Econ Finance. 2017;40:159–175.
[6] Faraci F, Jadamba B, Raciti F. On stochastic variational inequalities with mean value constraints. J Optim Theory Appl. 2016;171(2):675–693.
[7] Gwinner J, Raciti F. On a class of random variational inequalities on random sets. Numer Funct Anal Optim. 2006;27:619–636.
[8] Jadamba B, Khan AA, Raciti F. Regularization of stochastic variational inequalities and a comparison of an Lp and a sample-path approach. Nonlinear Anal. 2014;94:65–83.
[9] Pappalardo M, Passacantando M, Raciti F. A stochastic network equilibrium model for electric power markets with uncertain demand. Optimization. 2020;69:1703–1730.
[10] King AJ, Rockafellar RT. Sensitivity analysis for nonsmooth generalized equations. Math Program. 1992;55:193–212.
[11] King AJ, Rockafellar RT. Asymptotic theory for solutions in generalized M-estimation and stochastic programming. Math Oper Res. 1990;18:148–162.
[12] Lamm M, Lu S. Generalized conditioning based approaches to computing confidence intervals for solutions to stochastic variational inequalities. Math Program. 2019;174:99–127.
[13] Lamm M, Lu S, Budhiraja A. Individual confidence intervals for solutions to expected value formulations of stochastic variational inequalities. Math Program. 2017;165:151–196.
[14] Lu S. A new method to build confidence regions for solutions of stochastic variational inequalities. Optimization. 2014;63:1431–1443.

[15] Lu S. Symmetric confidence regions and confidence intervals for normal map formulations of stochastic variational inequalities. SIAM J Optim. 2014;24:1458–1484.
[16] Lu S, Budhiraja A. Confidence regions for stochastic variational inequalities. Math Oper Res. 2013;38:545–568.
[17] Lu S, Liu Y, Yin L, et al. Confidence intervals and regions for the lasso by using stochastic variational inequality techniques in optimization. J R Stat Soc: Ser B. 2017;79:589–611.
[18] Owen AB. Empirical likelihood ratio confidence intervals for a single functional. Biometrika. 1988;75:237–249.
[19] Pang JS. Error bound in mathematical programming. Math Program. 1997;79:299–332.
[20] Lam H, Zhou E. The empirical likelihood approach to quantifying uncertainty in sample average approximation. Oper Res Lett. 2017;45:301–307.
[21] Lam H. Robust sensitivity analysis for stochastic systems. Math Oper Res. 2016;41:1248–1275.
[22] Duchi J, Glynn P, Namkoong H. Statistics of robust optimization: a generalized empirical likelihood approach. arxiv, 2016.
[23] Ben-Tal A, Hertog D, Waegenaere AD, et al. Robust solutions of optimization problems affected by uncertain probabilities. Manage Sci. 2013;59:341–357.
[24] Liu Y, Xu H. Entropic approximation for mathematical programs with robust equilibrium constraints. SIAM J Optim. 2014;24:933–958.
[25] Grant M, Boyd S. CVX, for convex optimization, http://www.stanford.edu/boyd/.
[26] Pflug GC. Stochastic optimization and statistical inference, In: Ruszczynski A, Shapiro A, editos. Stochastic programming. Amsterdam: Elsevier; 2003. p. 483–554. (Handbooks in Operations Research and Management Science; vol. 10).
[27] Guigues V, Juditsky A, Nemirovski A. Non-asymptotic confidence bounds for the optimal value of a stochastic program. Optim Methods Softw. 2017;32:1033–1058.
[28] Tassa Y. LCP Solver. https://www.cs.washington.edu/people/postdocs/tassa/code/.
[29] Yu G, Yin L, Lu S, et al. Confidence intervals for sparse penalized regression with random designs. J Am Stat Assoc. 2020;115:794–809.