变分分析–基础理论与前沿进展

稳定性的变分准则

第1讲：稳定性及变分准则

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1) 最小二乘的LM

$$F(x)=0 \iff \min \frac{1}{2}\|F(x)\|^2$$

Newton $\left(\underbrace{JF(x^k)^T JF(x^k)} + \lambda_k I\right) d = r^k.$

解误差条件 ✓

2) 邻近点方法

$$0 \in T(x), \quad x^{k+1} \approx P_k(x^k), \quad P_k = (I + c_k T)^{-1}$$

Rockafellar (1976) $x^k \to \bar{x}, \quad \bar{x} \in T^{-1}(0)$

收敛速度：T^{-1} 在 $(0, \bar{x})$ 处弱满平稳性.

3) ADMM 收敛速度：误差界条件

目录

素材基于

- Bonnans J. F. and Shapiro A., *Perturbation Analysis of Optimization Problems*, Springer-Verlag, New York, 2000.
- Rockafellar R.T., *Convex Analysis*, Princeton University Press, 1970.
- Rockafellar R.T. and Wets R.J.-B, *Variational Analysis*, Springer-Verlag, New York, 1998.
- 张立卫,殷子然,最优化问题的稳定性分析,科学出版社,2020.

稳定性概念

Berge半连续性

定义 1.1

$u \in U,\ S(u) \subseteq Y$

设U, Y是度量空间, 对于集值映射$S : U \rightrightarrows Y$, $S: u \to 2^Y$

(1) 称集值映射S在$u_0 \in U$处是Berge上半连续的, 若对每一个满足$S(u_0) \subset \mathcal{O}$的开集合$\mathcal{O}$, 存在$\delta > 0$, 满足

$$S(u) \subset \mathcal{O}, \forall u \in \mathbb{B}_\delta(u_0).$$

(2) 称集值映射S在$u_0 \in X$处是Berge下半连续的, 若对每一个满足$S(u_0) \cap \mathcal{O} \neq \emptyset$的开集合$\mathcal{O}$, 存在$\delta > 0$, 满足

$$S(u) \cap \mathcal{O} \neq \emptyset, \forall u \in \mathbb{B}_\delta(u_0).$$

Hausdorff半连续性

定义 1.2

设U, Y是度量空间, 对于集值映射$S : U \rightrightarrows Y$,

(1) 称集值映射S在$u_0 \in U$处是*Hausdorff* 意义下上半连续的, 若任意$\varepsilon > 0$, $\exists \delta > 0$, 满足

$$S(u) \subset S(u_0) + \varepsilon\mathbb{B}, \forall u \in \mathbb{B}_\delta(u_0).$$

(2) 称集值映射S在$u_0 \in U$处是*Hausdorff* 意义下下半连续的, 若任意 $\varepsilon > 0$, $\exists \delta > 0$, 满足

$$S(u_0) \subset S(u) + \varepsilon\mathbb{B}, \forall u \in \mathbb{B}_\delta(u_0).$$

外半连续，内半连续

$$\limsup_{u\to u_0} S(u) \subseteq S(u_0) \quad (等号)$$

$$\liminf_{u\to u_0} S(u) \supseteq S(u_0) \quad (S\ 是闭值，等号)$$

$$\limsup_{u\to u_0} S(u) = \{x : \exists u^k \to u^0,\ \exists x^k \in S(u^k),\ x^k \to x\}$$

$$\liminf_{u\to u_0} S(u) = \{x : \forall u^k \to u^0,\ \exists x^k \in S(u^k),\ x^k \to x\}$$

Hausdorff 距离

A、C

$$d_\infty(A,C) = \inf\{\varepsilon>0 : \underline{A \subseteq C+\varepsilon\mathbb{B},}\ C \subseteq A+\varepsilon\mathbb{B}\}$$

$$\varepsilon_1 = \inf\{\varepsilon>0 : A \subseteq C+\varepsilon\mathbb{B}\}$$

$$\varepsilon_2 = \inf\{\varepsilon\geq 0,\ C \subseteq A+\varepsilon\mathbb{B}\}$$

$$d_\infty(A,C) = \max\{\varepsilon_1, \varepsilon_2\}$$

$S(u_0)$ $S(u)$

$d_\infty(A,C) \leq \varepsilon \iff A \subseteq C+\varepsilon\mathbb{B} \longrightarrow$ 下半连续

$C \subseteq A+\varepsilon\mathbb{B} \longrightarrow$ 上半连续

$S(u)$ $S(u_0)$

$x^1, x^2 \in x + \delta \mathbb{B}$, $\|F(x^1) - F(x^2)\| \le L\|x^1 - x^2\|$

一个例子

设$F : \Re^n \to \Re^m$在$x \in \Re^n$处局部Lipschitz连续, $\partial F(x)$是Clarke广义Jacobian

几乎处处可微的

$$\partial F(x) = \mathrm{conv}\{\lim \mathcal{J} F(x_i) : x_i \to x, x_i \notin \Omega_F\}.$$

Ω_F：不可微点全体

由[4, 2.6.2 Proposition][1],

(a) $\partial F(x)$是$\Re^{m\times n}$的一非空闭紧致集合.

(b) ∂F在x处是闭的;即$x_t \to x$, $Z_t \in \partial F(x_t)$, $Z_t \to Z$, 则$Z \in \partial F(x)$.

外半连续的

(c) ∂F在x处是上半连续的:$\forall \epsilon > 0$, $\exists \delta > 0$满足

Hausdorff意义 下上半连续

$$\partial F(y) \subset \partial F(x) + \epsilon \mathbf{B}_{m\times n}, \ \forall y \in x + \delta \mathbf{B}.$$

[1] Clarke F H. *Optimization and Nonsmooth Analysis*. New York: John Wiley and Sons, 1983.

Berge与Huasdorff半连续性的关系

Berge意义下的连续性和 Hausdorff意义下的连续性有如下的关系,见[1].[2]

(a) 如果集值映射S在u_0处Hausdorff意义下上半连续且$S(u_0)$是紧致的, 则S在u_0处Berge意义下上半连续.

(b) 如果集值映射S在u_0处Berge 意义下上半连续, 则S在u_0处Hausdorff意义下上半连续.

(c) 如果集值映射S在u_0处Hausdorff意义下下半连续, 则S在u_0处Berge意义下下半连续.

(d) 如果集值映射S在u_0处Berge 意义下下半连续且$\mathrm{cl}S(u_0)$是紧致的, 则S在Hausdorff意义下是下半连续的.

[2] Bank B, Guddat J, Klatte D, Kummer B and Tammer K. *Nonlinear Parametric Optimization*. Berlin: Akademie-Verlag, 1982.

与外半连续性的关系

对 $u^k \to u_0$, $x^k \in S(u^k)$, $x^k \to x_0 \Rightarrow x_0 \in S(u_0)$

闭性.

"在局部有界的条件下", 集合的外半连续性等价于Berge意义下的上半连续的[24, 定理 5.19].[3] 对闭值的集值映射, Hausdorff意义下上半连续等价于外半连续性(闭性)[24, 命题5.12(a)].

局部有界: S 在 u_0 处局部有界

$\exists \delta > 0$, $\exists$ 集合 C, C 有界

$\forall u \in u_0 + \delta \mathbb{B}$, $S(u) \subseteq C$.

[3] Rockafellar R T and Wets R J B. *Variational Analysis*. New York: Springer-Verlag, 1998.

定义 1.3

集值映射$F : \mathcal{X} \rightrightarrows \mathcal{Y}$在$(x^0, y^0) \in \operatorname{gph} F$处是以率$\kappa$为度量正则的, 如果存在邻域$U \in \mathcal{N}(x^0)$, $V \in \mathcal{N}(y^0)$和常数$\kappa > 0$, 满足

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)), \ \forall (x, y) \in U \times V.$$

定义 1.4

集值映射$F : \mathcal{X} \rightrightarrows \mathcal{Y}$在$x^0$对$y^0$是度量次正则的, 如果$(x^0, y^0) \in \operatorname{gph} F$, 存在邻域$U \in \mathcal{N}(x^0)$, $V \in \mathcal{N}(y^0)$和常数$\kappa > 0$, 满足

$$d(x, F^{-1}(y^0)) \leq \kappa d(y^0, F(x) \cap V), \forall x \in U.$$

$\min f(x) \quad s.t. \quad G(x) \in K$

一个例子

考虑约束集合

$\Phi(y) = \{x \in \mathcal{X} : G(x) + y \in K\}, \quad \Phi = \Phi(0)$

约束系统

$$\Phi = \{x \in \mathcal{X} : G(x) \in K\}, \qquad G : \mathcal{X} \to \mathcal{Y}$$

其中$\mathcal{X}$与$\mathcal{Y}$是有限维Hilbert空间,$K \subset \mathcal{Y}$是非空闭凸集合. 定义

$$\underline{\mathcal{F}_G(x) = K - G(x)},$$

则$\Phi = \mathcal{F}_G^{-1}(0)$. 设$x_0 \in \Phi$, 则 $\mathcal{F}_G$在$(x_0, 0)$处满足度量正则性意味着

$\underline{\mathrm{dist}(x, \Phi)} = \underline{\mathrm{dist}(x, \mathcal{F}_G^{-1}(0))} \leq \underline{\kappa\,\mathrm{dist}(0, \mathcal{F}_G(x))}$

$$\underline{\mathrm{dist}\,(x, \mathcal{F}_G^{-1}(y))} \leq \underline{\kappa \mathrm{dist}\,(y, \mathcal{F}_G(x))} = \underline{\kappa \mathrm{dist}\,(G(x) + y, K)}$$

对$(x, y) \in \mathbf{B}_\delta(x_0) \times \delta\mathbf{B}$成立.

定义 1.5

(*Aubin*性质和图模). 称集值映射$S:\mathcal{X}\rightrightarrows\mathcal{Y}$相对于$X$在$x^0$点关于$u^0$具有*Aubin*性质, 其中$x^0\in X$, $u^0\in S(x^0)$, 若$\mathrm{gph}S$在(x^0,u^0)点处是局部闭的, 且存在邻域$V\in\mathcal{N}(x^0)$, $W\in\mathcal{N}(u^0)$ 和常数$\kappa\in\Re_+$, 满足

$$S(x')\cap W\subset S(x)+\kappa\|x'-x\|\boldsymbol{B},\ \forall x,x'\in X\cap V. \tag{1}$$

若将上述条件中的$X\cap V$替换为V, 则称*Aubin*性质在x^0点关于u^0成立. 此时, S在x^0点关于u^0的图模*(graphical modulus)*为

$$\mathrm{lip}S(x^0|u^0):=\inf\{\kappa:\exists V\in\mathcal{N}(x^0),W\in\mathcal{N}(u^0),\ \text{满足}\ S(x')\cap W\subset S(x)+\kappa\|x'-x\|\boldsymbol{B},\ \forall x,x'\in V\}.$$

定义 1.6

称集值映射$F:\mathcal{X} \rightrightarrows \mathcal{Y}$在$x^0$处关于$y^0$是平稳的*(calm)*, 如果$y^0 \in F(x^0)$, 存在一常数$\kappa > 0$, x_0 的一邻域V 和 y^0 的一邻域W满足

$$F(x) \cap W \subseteq F(x^0) + \kappa \|x - x^0\| \boldsymbol{B}, \quad \forall x \in V.$$

定义 1.7

称集值映射$F:\mathcal{X} \rightrightarrows \mathcal{Y}$在$x^0$处关于$y^0$是稳健平稳的*(robustly calm)*, 如果$y^0 \in F(x^0)$, 存在一常数$\kappa > 0$, x_0 的一邻域V 和 y^0 的一邻域W满足

$$\emptyset \neq F(x) \cap W \subseteq F(x^0) + \kappa \|x - x^0\| \boldsymbol{B}, \quad \forall x \in V.$$

定义 1.8

称集值映射$F:\mathcal{X}\rightrightarrows\mathcal{Y}$在$x^0$处关于$y^0$是孤立平稳的*(isolated calm)*, 如果$y^0\in F(x^0)$, 存在一常数$\kappa>0$, x_0 的一邻域V 和 y^0 的一邻域W满足

$$F(x)\cap W\subseteq\{y^0\}+\kappa\|x-x^0\|\mathbb{B},\quad\forall x\in V.$$

定义 1.9

称集值映射$F:\mathcal{X}\rightrightarrows\mathcal{Y}$在$x^0$处关于$y^0$是稳健孤立平稳的*(robustly isolated calm)*, 如果$y^0\in F(x^0)$, 存在一常数$\kappa>0$, x_0 的一邻域V和 y^0的一邻域W 满足

$$\emptyset\neq \underline{F(x)\cap W}\subseteq\{y^0\}+\kappa\|x-x^0\|\mathbb{B},\quad\forall x\in V.$$

定义 1.10

称集值映射$F : \mathcal{X} \rightrightarrows \mathcal{Y}$在$x^0$处是上*Lipschitz* 的,如果存在一常数$\kappa > 0$和$x_0$ 的一邻域V 满足

$$F(x) \subseteq F(x^0) + \kappa \|x - x^0\| \boldsymbol{B}, \quad \forall x \in V.$$

定义 1.11

称集值映射$F : \mathcal{X} \rightrightarrows \mathcal{Y}$在$x^0$处关于$y^0$是线性开的,如果存在邻域$V \in \mathcal{N}(x^0)$, $W \in \mathcal{N}(y^0)$ 与常数 $\kappa \in \Re_+$, 满足

$$F(x + \kappa\varepsilon \boldsymbol{B}) \supset [F(x) + \varepsilon \boldsymbol{B}] \cap W, \ \forall x \in V, \varepsilon > 0;$$

$\Psi: X \rightrightarrows Y$ 在 $(x_0, y_0) \in \operatorname{gph} \Psi$ 以线性率 $\gamma > 0$ 为开的，若 $\exists\, t_{max} > 0$，及

(x_0, y_0) 的一邻域 V，

$$\forall (x,y) \in \operatorname{gph} \Psi \cap V, \ \forall t \in [0, t_{max}]$$

$$y + t\gamma \mathbb{B}_Y \subseteq \Psi(x + t\mathbb{B}_X)$$

定义 1.12

(集值映射的*Lipschitz*连续性). 称映射$S:\Re^n \rightrightarrows \Re^m$在$\Re^n$的子集$X$上是*Lipschitz* 连续的, 若它在$X$上是非空闭值的且存在$\kappa \in \Re_+$为*Lipschitz* 常数, 满足

$$\mathbf{d}_\infty(S(x'), S(x)) \le \kappa\|x' - x\|, \ \forall x, x' \in X,$$

或等价地, $d_\infty(A,B) = \inf\{\varepsilon: A \subseteq B + \varepsilon \mathbb{B},\ B \subseteq A + \varepsilon \mathbb{B}\}$

$$S(x') \subset S(x) + \kappa\|x' - x\|\mathbf{B}, \ \forall x, x' \in X.$$

隐函数定理推广

$0 = g(x,p)$, $x(p)$

强正则性

$\delta = g(x_0,p_0) + D_x g(x_0,p_0)(x-x_0)$

$x(\delta)$

定义 1.13

考虑广义方程

$$0 \in f(x,p) + F(x),$$

其中$F : \mathcal{X} \rightrightarrows \mathcal{Y}$ 是一集值映射. 定义

$$G(x) = f(x^0,p^0) + D_x f(x^0,p^0)(x-x^0) + F(x).$$

如果G^{-1}是从$0 \in \mathcal{Y}$的一个邻域到x^0的一邻域的单值*Lipschitz*连续映射, 则称广义方程在(x^0,p^0)处是强正则的.

$\delta \in G(x) \iff x(\cdot): \varepsilon \mathbb{B}_Y \to x^0 + \varepsilon_1 \mathbb{B}_X$ 单值, Lip. 连续

Karush-Kuhn-Tucker系统

考虑约束优化问题

$N_K(y) = K^\circ \cap y^\perp$

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & G(x) \in K, \end{aligned} \tag{2}$$

$v \in N_K(y) \Leftrightarrow y \in N_{K^\circ}(v)$

其中$f : X \to \Re$, $G : X \to Y$, $K \subset Y$是一闭凸锥, X, Y是有限维的Hilbert空间. Karush-Kuhn-Tucker条件

$$\mathrm{D}_x L(x, \lambda) = 0, \ \lambda \in N_K(G(x)).$$

$\Leftrightarrow G(x) \in N_{K^\circ}(\lambda)$

写成广义方程形式

$$\text{“} 0 \in F(x, \lambda) + N_{\Re^n \times K^\circ}(x, \lambda) \text{”}, \ F(x, \lambda) = \begin{bmatrix} \mathrm{D}_x L(x, \lambda) \\ -G(x) \end{bmatrix}.$$

参数Karush-Kuhn-Tucker系统

考虑参数约束优化问题

$$
\begin{array}{ll}
\min\limits_{x} & f(x,u) \\
\text{s.t.} & G(x,u) \in K,
\end{array}
\tag{3}
$$

其中$f : X \times U \to \Re$, $G : X \times U \to Y$, $K \subset Y$是一闭凸锥, X, Y是有限维的Hilbert空间. Karush-Kuhn-Tucker条件

$$
\mathrm{D}_x L(x,u,\lambda) = 0,\ \lambda \in N_K(G(x,u)).
$$

写成广义方程形式

$$
0 \in F(x,u,\lambda) + N_{\Re^n \times K^\circ}(x,\lambda),\ F(x,u,\lambda) = \begin{bmatrix} \mathrm{D}_x L(x,u,\lambda) \\ -G(x,u) \end{bmatrix}.
$$

强正则性

$$\delta \in F(x_0, u_0, \lambda_0) + \mathrm{D}_x F(x_0, u_0, \lambda_0) \begin{bmatrix} (x - x_0) \\ (\lambda - \lambda_0) \end{bmatrix} + N_{\Re^n \times K^\circ}(x, \lambda)$$

在$(0, x_0, \lambda_0)$附近有唯一的Lipschitz连续的映射$(x(\delta), \lambda(\delta))$.

系统详细写成:

$$\begin{bmatrix} \delta_x \\ \delta_\lambda \end{bmatrix} \in \begin{bmatrix} \mathrm{D}_x L(x_0, u_0, \lambda_0) \\ -G(x_0, u_0) \end{bmatrix} + \begin{bmatrix} \mathrm{D}^2_{xx} L(x_0, u_0, \lambda_0) & \mathrm{D}_x G(x_0, u_0)^* \\ -\mathrm{D}_x G(x_0, u_0) & 0 \end{bmatrix} \begin{bmatrix} (x - x_0) \\ (\lambda - \lambda_0) \end{bmatrix} + N_{\Re^n \times K^\circ}(x, \lambda).$$

$\delta = \underset{\sim}{H}(u)$

Lipschitz同胚

定义 1.14

(局部 *Lipschitz* 同胚). 称连续函数 $F : \mathcal{O} \subseteq \mathcal{X} \to \mathcal{X}$在 $x \in \mathcal{O}$ 处是局部 *Lipschitz* 可逆的, 如果存在x的一个开邻域 $\mathcal{N} \subseteq \mathcal{O}$使得限定在这个邻域上的映射 $F|_{\mathcal{N}} : \mathcal{N} \to F(\mathcal{N})$ 是双射并且它的逆函数是 *Lipschitz* 连续的. 称F在x附近是局部 *Lipschitz* 同胚的, 如果F在x附近是局部 *Lipschitz* 可逆的并且F在x处是局部 *Lipschitz* 连续的.

一般形式的约束优化问题

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & G(x) \in K, \end{aligned} \tag{4}$$

其中$f : X \to \Re$, $G : X \to Y$, $K \subset Y$是一闭凸集合, X, Y是有限维的Hilbert空间. 设U是一Banach空间, $f : X \times U \to \Re$, $G : X \times U \to Y$.

$\mathcal{C}^2$-光滑参数化

称$(f(x,u), G(x,u))$, $u \in U$, 是问题(4)的一$\mathcal{C}^2$-光滑参数化, 如果$f(\cdot,\cdot)$与$G(\cdot,\cdot)$是二次连续可微的, 且存在$\overline{u} \in U$满足$f(\cdot,\overline{u}) = f(\cdot)$, $G(\cdot,\overline{u}) = G(\cdot)$. 相对应的参数优化问题具有下述形式

$$\begin{array}{ll} \min\limits_{x} & f(x,u) \\ \text{s.t.} & G(x,u) \in K. \end{array} \tag{5}$$

称上述参数化是标准的(canonical), 如果$U := X \times Y$, $\overline{u} = (0,0) \in X \times Y$, 且

$$\begin{aligned} (f(x,u), G(x,u)) = \ & (f(x) - \langle u_1, x\rangle, G(x) + u_2), \\ & x \in X,\ u = (u_1, u_2) \in X \times Y. \end{aligned}$$

强稳定性

现在介绍[2, Definition 5.33][4] 中的稳定点的强稳定性的概念，它在优化问题的灵敏度分析中起重要的作用.

定义 1.15

设x^*是问题(4)的稳定点. 称在x^*处关于$\mathcal{C}^2$-光滑参数化$(f(x,u), G(x,u))$是强稳定的*(strongly stable)*，如果存在 x^*的邻域$\mathcal{V}_X$与$\bar{u}$的邻域$\mathcal{V}_U \subset U$，满足对任何$u \in \mathcal{V}_U$，问题(5)存在唯一的稳定点$x(u) \in \mathcal{V}_X$，$x(\cdot)$在$\mathcal{V}_U$ 上连续. 如果这一性质对每一$\mathcal{C}^2$-光滑参数化均是成立的，则称x^*是强稳定的.

[4] Bonnans J F and Shapiro A. *Perturbation Analysis of Optimization Problems*. New York: Springer-Verlag, 2000.

下述一致二阶增长条件的定义取自[2, Definition 5.16].

定义 1.16

设x^*是问题(4)的稳定点. 称在x^*处关于C^2-光滑参数化$(f(x,u), G(x,u))$的一致二阶增长条件成立, 如果存在$\alpha > 0$, x^*的邻域$\mathcal{V}_X$与$\bar{u}$的邻域$\mathcal{V}_U \subset U$, 满足对任何$u \in \mathcal{V}_U$与问题(5)的稳定点$x(u) \in \mathcal{V}_X$, 下述不等式成立:

$$f(x,u) \geq f(x(u),u) + \alpha \|x - x(u)\|^2, \ \forall x \in \mathcal{V}_X \text{满足} G(x,u) \in K. \tag{6}$$

称在x^*处的一致二阶增长条件成立, 如果(6)式对问题(4)的任何C^2-光滑参数化均是成立的.

定理 1.1

[24, Theorem 9.43][5] 若gph$S$在$(\bar{x},\bar{u})\in \mathrm{gph}S$点处是局部闭的, 则下述条件等价:

(a) (逆Aubin性质): S^{-1} 在$\bar{u}$点关于$\bar{x}$具有Aubin性质;

(b) (度量正则性): $\exists V\in\mathcal{N}(\bar{x}),\ W\in\mathcal{N}(\bar{u}),\ \kappa\in\Re_+$, 满足

$$d(x,S^{-1}(u))\le \kappa d(u,S(x)),\ \text{若} x\in V, u\in W;$$

(c) (线性开性): $\exists V\in\mathcal{N}(\bar{x}),\ W\in\mathcal{N}(\bar{u}),\ \kappa\in\Re_+$, 满足

$$S(x+\kappa\varepsilon \boldsymbol{B})\supset [S(x)+\varepsilon\boldsymbol{B}]\cap W,\ \ \forall x\in V,\varepsilon>0;$$

(d) (伴同导数非奇异性): 满足$0\in D^*S(\bar{x}|\bar{u})(y)$的$y$只有$y=0$.

$\Updownarrow$

$D^*S^{-1}(\bar{u}|\bar{x})(0)=\{0\}$

[5]Rockafellar R T and Wets R J B. *Variational Analysis*. New York: Springer-Verlag, 1998.

度量次正则性与平稳性

下述定理表明集值映射的度量次正则性等价于逆映射的平稳性.

定理 1.2

[7, Theorem 3H.3][6] 设集值映射$F:\Re^n \rightrightarrows \Re^m$, $\bar{y}\in F(\bar{x})$. 那么F在$\bar{x}$处关于$\bar{y}$以常数$\kappa>0$是度量次正则的当且仅当它的逆映射$F^{-1}:\Re^m \rightrightarrows \Re^n$在$\bar{y}$处关于$\bar{x}$以相同的常数$\kappa>0$是平稳的, 并且有$\mathrm{clm}(F^{-1};\bar{y}|\bar{x})=\mathrm{subreg}(F;\bar{x}|\bar{y})$.

[6] Dontchev A L and Rockafellar R T. *Implicit Functions and Solution Mappings*. New York: Springer, 2009.

稳定性的变分准则

非凸集合的切锥

设$C \subset \Re^n$是一闭集合.

$T_C(\bar{x}) = \{d : \exists t_k, \ dist(\bar{x}+t_k d, C) = o(t_k)\}$

- Radial cone/雷达锥

$$\mathcal{R}_C(\overline{x}) = \{d : \exists t^* > 0, \forall t \in [0, t^*], \overline{x} + td \in C\}.$$

- Tangent cone/切锥

$T_C(\bar{x}) = \limsup_{t \searrow 0} \frac{C - \bar{x}}{t}$

$$T_C(\overline{x}) = \{d : \exists t_k \searrow 0, d^k \to d, \underline{\overline{x}} + t_k d^k \in C\}.$$

- Regular tangent cone/正则切锥

$$\widehat{T}_C(\overline{x}) = \{d : \forall y^k \to \overline{x}, \forall t_k \searrow 0, \exists d^k \to d, \underline{y}^k + t_k d^k \in C\}.$$

$\hat{T}_C(\bar{x}) = \liminf_{\substack{t \searrow 0 \\ x \to \bar{x}}} \frac{C - x}{t}$

非凸集合的法锥

- Regular normal cone/正则法锥

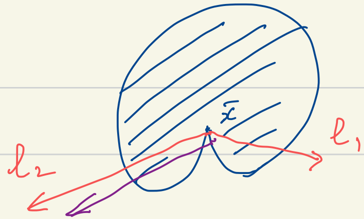
$$\widehat{N}_C(\bar{x}) = \{v \in \Re^n : \langle v, x - \bar{x}\rangle \leq \mathrm{o}(|x - \bar{x}|) \text{ for } x \in C\}.$$

- Normal cone/法锥

$$N_C(\bar{x}) = \left\{ v \in \Re^n : \begin{array}{l} \exists x^k \in C, \exists v^k \in \widehat{N}_C(x^k) \\ \text{such that } v^k \to v \end{array} \right\}.$$

- 切法共轭关系

1)



$\hat{N}_C(\bar{x}) = \{0\}$，$\hat{N}_C$ 不稳定的

$N_C(\bar{x}) = \{0, l_1, l_2\}$，$N_C$ 稳定的

无用

2) $C = \{x \in X : F(x) \in D\}$，$\hat{T}_C(\bar{x}) \supseteq \{d \in \hat{T}_X(\bar{x}) : JF(\bar{x})\, d \in \hat{T}_D(F(\bar{x}))\}$，

$0 \in JF(\bar{x})^T y \in N_D(F(\bar{x})) + N_X(\bar{x}) \implies y = 0$ 基本CQ

$\implies T_C(\bar{x}) \subseteq \{d \in T_X(\bar{x}) : JF(\bar{x})\, d \in T_D(F(\bar{x}))\}$

切[illegible]关系

$$
\begin{tikzcd}
T_C \arrow[r, "*"] \arrow[d, "\liminf\limits_{x \xrightarrow{C} x^0}"] & \hat{N}_C \arrow[d, "\limsup\limits_{x \xrightarrow{C} x^0}"] \\
\hat{T}_C & N_C \arrow[l, "*"']
\end{tikzcd}
$$

(有穷维空间)

Graphical derivatives and coderivatives

定义 2.1

The graphical derivative of S at $\bar{x}$ for any $\bar{u} \in S(\bar{x})$ is the mapping $DS(\bar{x}|\bar{u}) : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ defined by

$$z \in DS(\bar{x}|\bar{u})(w) \iff (w, z) \in T_{gph\ S}(\bar{x}, \bar{u}),$$

whereas the coderivative is $D^*S(\bar{x}|\bar{u}) : \mathbf{R}^{\overset{m}{n}} \rightrightarrows \mathbf{R}^{\overset{n}{\cancel{m}}}$ defined by

$$v \in D^*S(\bar{x}|\bar{u})(y) \iff (v, -y) \in N_{gph\ S}(\bar{x}, \bar{u}).$$

$$\Delta_\tau S(\bar{x}|\bar{u})(w) = \frac{1}{\tau}\left[S(\bar{x}+\tau w) - \bar{u} \right]$$

$$H = g\text{-}\limsup_{\tau \downarrow 0} \Delta_\tau S(\bar{x}|\bar{u}) \iff \operatorname{gph} H = \limsup_{\tau \downarrow 0} \operatorname{gph} \Delta_\tau S(\bar{x},\bar{u})$$

$$\operatorname{gph} \Delta_\tau S(\bar{x},\bar{u}) = \operatorname{gph} \frac{1}{\tau}\left[S(\bar{x}+\tau \cdot) - \bar{u} \right] = \frac{1}{\tau}\left[\operatorname{gph} S - (\bar{x},\bar{u}) \right]$$

$$\operatorname{gph} H = \limsup_{\tau \downarrow 0} \frac{1}{\tau}\left[\operatorname{gph} S - (\bar{x},\bar{u}) \right] = T_{\operatorname{gph} S}(\bar{x},\bar{u})$$

$$H = DS(\bar{x}|\bar{u}).$$

$z \in DS(\bar{x}|\bar{u})(w) \implies z = ?$ $\quad (w,z) \in T_{\operatorname{gph} S}(\bar{x},\bar{u})$

$\iff \exists\, t_k \downarrow 0,\ w^k \to w,\ z^k \to z$

s.t. $(\bar{x}+t_k w^k,\ \bar{u}+t_k z^k) \in \operatorname{gph}(S)$

$\iff \bar{u} + t_k z^k \in S(\bar{x}+t_k w^k)$

$\iff z^k \in \dfrac{S(\bar{x}+t_k w^k) - \bar{u}}{t_k}$

$$DS(\bar{x}|\bar{u})(w) = \limsup_{\substack{w' \to w \\ t \downarrow 0}} \frac{S(\bar{x}+t w') - \bar{u}}{t}$$

Regular graphical derivatives and coderivatives

定义 2.2

The regular derivative $\widehat{D}(\bar{x}|\bar{u}) : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ *and the regular coderivative* $\widehat{D}^*(\bar{x}|\bar{u}) : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ *are defined by*

$$z \in \widehat{D}S(\bar{x}|\bar{u})(w) \Longleftrightarrow (w, z) \in \widehat{T}_{gph\ S}(\bar{x}, \bar{u}),$$
$$v \in \widehat{D}^*S(\bar{x}|\bar{u})(y) \Longleftrightarrow (v, -y) \in \widehat{N}_{gph\ S}(\bar{x}, \bar{u}).$$

Strict graphical derivatives

For a mapping $S : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, the strict derivative mapping $D_*S(\bar{x}|\bar{u}) : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ for S at $\bar{x}$ for $\bar{u}$, where $\bar{u} \in S(\bar{x})$, is defined by

$u^\nu \in S(x^\nu)$

$$D_*S(\bar{x}|\bar{u})(w) := \Big\{ z \,|\, \exists \underline{\tau}^\nu \searrow 0, (\underline{x^\nu, u^\nu}) \overset{\mathrm{gph}\, S}{\to} (\underline{\bar{x}}, \underline{\bar{u}}), w^\nu \to w, \\ \text{with } z^\nu \in [S(x^\nu + \tau^\nu w^\nu) - u^\nu]/\tau^\nu, z^\nu \to z \Big\}$$

or in other words, for $\Delta_\tau S(x|u)(w) := \dfrac{S(x+\tau w) - u}{\tau}$

$$D_*S(\bar{x}|\bar{u}) := \text{g-limsup}_{\tau \searrow 0, (x,u) \overset{\mathrm{gph}\, S}{\to} (\bar{x}, \bar{u})} \Delta_\tau S(x|u).$$

Note that if F is strictly continuous, one then has the simpler formula

$$D_*F(\bar{x})(w) = \Big\{ z | \exists \tau^\nu \searrow 0, x^\nu \to \bar{x} \text{ with } \Delta_{\tau^\nu} F(x^\nu)(w) \to z \Big\}.$$

Mordukhovich准则

Mordukhovich准则提供了一种计算公式, 可用于判断集值映射是否具有Aubin性质.

$S(x') \cap W \subseteq S(x) + \kappa \|x' - x\| \mathbb{B}$

描述性

定理 2.1

$x', x \in V$,

[24, Theorem 9.40] (*Mordukhovich准则*). 设$S : \Re^n \rightrightarrows \Re^m$, $\bar{x} \in \mathrm{dom} S$, $\bar{u} \in S(\bar{x})$. 设gph$S$ 在$(\bar{x}, \bar{u})$ 点是局部闭的. 则S在$\bar{x}$点关于$\bar{u}$具有*Aubin*性质当且仅当

$$D^* S(\bar{x}|\bar{u})(0) = \{0\},$$

或等价的, $|D^* S(\bar{x}|\bar{u})|^+ < \infty$.

$\Leftrightarrow (v, -y) \in N_{\mathrm{gph} S}(\bar{x}, \bar{u})$

$v \in D^* S(\bar{x}|\bar{u})(y)$

类比理解Mordukhovich准则

F is strictly continuous at $\bar{x}$ relative to X if $\bar{x} \in X$ and the value

$$\operatorname{lip}_X F(\bar{x}) := \limsup_{x,x' \xrightarrow{X} \bar{x},\, x \neq x'} \frac{|F(x') - F(x)|}{|x' - x|}$$

is finite.

9.13 Theorem(subdifferential characterization of strict continuity). *Suppose $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is locally lsc at $\bar{x}$ with $f(\bar{x})$ finite. Then the following conditions are equivalent:*

(a) *f is strictly continuous at $\bar{x}$,*

(b) $\partial^\infty f(\bar{x}) = \{0\}$;   horizon 次微分

(c) *the mapping $\widehat{\partial} f : x \to \widehat{\partial} f(x)$ is locally bounded at $\bar{x}$,*

(d) *the mapping $\partial f : x \to \partial f(x)$ is locally bounded at $\bar{x}$.*

- f 在 $\bar{x}$ 处严格连续 $\iff \partial^\infty f(\bar{x}) = \{0\}$

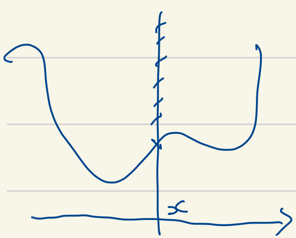
$$\partial^\infty f(\bar{x}) = \{(v, 0) \in N_{\mathrm{epi} f}(\bar{x}, f(\bar{x}))\}$$

$$(v, 0) \in N_{\mathrm{epi} f}(\bar{x}, f(\bar{x})) \implies v = 0$$

- Morduckhovich 准则

$$D^* S(\bar{x}|\bar{u})(y) = \{v : (v, -y) \in N_{\mathrm{gph} S}(\bar{x}, \bar{u})\}$$

$$D^* S(\bar{x}|\bar{u})(0) = \{0\}$$



$$(v, 0) \in N_{\mathrm{gph} S}(\bar{x}, \bar{u}) \implies v = 0.$$

$$\partial^\infty f(\bar{x}) = D^* E_f(\bar{x}|f(\bar{x}))(0)$$

- profile: $E_f(x) := \{\alpha : f(x) \le \alpha\}$, $\mathrm{gph}\, E_f = \mathrm{epi} f$,

强正则性–经典的隐函数定理的拓展

定理 2.2

[7] 设X是赋范向量空间,Y与Z是两个Banach空间, $\Omega \subset X \times Y$是包含点$(a, b)$的一开集合,$\phi \in \mathcal{C}(\Omega; Z)$ 满足

隐函数定理

(i) $\phi(a, b) = 0$;

(ii) $\dfrac{\partial \phi}{\partial y}(x, y) \in \mathcal{L}(Y; Z)$对任何$(x, y) \in \Omega$ 存在,且 $\dfrac{\partial \phi}{\partial y} \in \mathcal{C}(\Omega; \mathcal{L}(Y; Z))$;

(iii) $\dfrac{\partial \phi}{\partial y}(a, b) \in \mathcal{L}(Y; Z)$是双射, $\left(\dfrac{\partial \phi}{\partial y}(a, b)\right)^{-1} \in \mathcal{L}(Z, Y)$.

[7] pp.548-549, Philippe G. Ciarlet, Linear and Nonlinear Functional Analysis with Applications,SIAM,Philadelphia,2013.

$$f: \overset{a}{V} \to \overset{b}{W}$$

(a) 则存在X中的a的一开邻域V,存在Y中的b的一开邻域W与一隐函数$f \in \mathcal{C}(V; W)$满足

$$V \times W \subset \Omega, \left[\begin{array}{l} \{(x,y) \in V \times W : \phi(x,y) = 0\} \\ \quad = \{(x,y) \in V \times W : y = f(x)\}. \end{array} \right].$$

(b) 如果ϕ在$(a, b) \in \Omega$是可微的,那么f在a处是可微的,

$$f'(a) = -\left(\frac{\partial \phi}{\partial y}(a,b)\right)^{-1} \frac{\partial \phi}{\partial x}(a,b) \in \mathcal{L}(X; Y).$$

(c) 如果$\phi \in \mathcal{C}^m(\Omega; Z)$,其中$m \geq 1$是整数,则存在$X$中的$a$的一开邻域$\widetilde{V} \subset V$与$Y$中的$b$的一开邻域$\widetilde{W} \subset W$ 满足对任何$(x, y) \in \widetilde{V} \times \widetilde{W}$, $\frac{\partial \phi}{\partial y}(x, y) \in \mathcal{L}(Y; Z)$是双射,

$$\left(\frac{\partial \phi}{\partial y}(x, y)\right)^{-1} \in \mathcal{L}(Z, Y), f \in \mathcal{C}^m(\widetilde{V}; Y)$$

对任何$x \in \widetilde{V}$,

$$f'(x) = -\left(\frac{\partial \phi}{\partial y}(x, f(x))\right)^{-1} \frac{\partial \phi}{\partial x}(x, f(x)) \in \mathcal{L}(X; Y).$$

强正则性意味着什么?

考虑由参数广义方程定义的解映射

广义方程

$$S(x) = \{z \in \Re^k : 0 \in C(x,z) + N_Q(z)\}, \qquad (1)$$

其中$C : \mathcal{A} \times \Re^k \to \Re^k$是一连续可微映射, $\mathcal{A} \subset \Re^n$是一开集合, $Q \subset \Re^k$是一非空闭凸集合. 给定$(x_0, z_0) \in \mathrm{gph} S$, $x_0 \in \mathcal{A}$. **要寻求条件, 在这一条件下存在x_0的一个邻域$\mathcal{O}$和z_0的一个邻域V满足在$\mathcal{O}$上存在一个单值的Lipschitz连续映射$\sigma : \mathcal{O} \to V$使得**

$$\sigma(x_0) = z_0,\ \sigma(x) \in S(x),\ \forall x \in \mathcal{O}, \qquad (2)$$

或者使得

$$\sigma(x) = S(x) \cap V,\ \forall x \in \mathcal{O}. \qquad (3)$$

隐函数定理

$F(x, y) = 0$

$$\left.\begin{array}{l} ① \ F(\bar{x}, \bar{y}) = 0 \\ ② \ D_y F(\bar{x}, \bar{y}) \text{ 可逆的} \\ ③ \ F \text{ 在 } (\bar{x}, \bar{y}) \text{ 附近 } C^1 \end{array}\right\} \Rightarrow \begin{array}{l} \exists \delta > 0,\ \varepsilon > 0 \\ \exists y: \mathbb{B}_\delta(\bar{x}) \to \mathbb{B}_\varepsilon(\bar{y}) \\ F(x, y(x)) \equiv 0,\ \forall x \in \mathbb{B}_\delta(\bar{x}) \end{array}$$

$$\delta = F(\bar{x}, \bar{y}) + D_y F(\bar{x}, \bar{y})(y - \bar{y})$$

有解，$y(\delta)$ 连续可微的

与强正则性表述同

定义

$$\Sigma(\xi) = \{z \in \Re^k : \xi \in C(x_0, z_0) + \mathcal{J}_z C(x_0, z_0)(z - z_0) + N_Q(z)\} \tag{4}$$

与

$$r(x, z) = C(x_0, z_0) + \mathcal{J}_z C(x_0, z_0)(z - z_0) - C(x, z).$$

容易验证下述结论.

命题 2.1

下述关系成立

$S(x) = \{z : C(x, z) + N_Q(z)\}$

$$z \in S(x) \text{当且仅当} z \in \Sigma(r(x, z)).$$

证明

根据Σ与r的定义, 有

$$z \in S(x)\text{当且仅当 } 0 \in C(x,z) + N_Q(z) \tag{5}$$

与

$$\begin{aligned} & z \in \Sigma(r(x,z))\text{当且仅当} \\ & r(x,z) \in C(x_0,z_0) + \mathcal{J}_z C(x_0,z_0)(z-z_0) + N_Q(z). \end{aligned} \tag{6}$$

简单的计算可得(5)与(6)中的广义方程是等价的. ■

由于C在$\mathcal{A} \times \Re^k$上是连续可微的, 可以选取x_0的邻域$\widetilde{U}$,z_0的邻域$\widetilde{V}$与一正的实常数L满足

$$\|C(x_1, z) - C(x_2, z)\| \leq L\|x_1 - x_2\|, \quad \forall x_1 \in \widetilde{U}, z \in \widetilde{V}. \tag{7}$$

x_2

定理 2.3

(a) 设存在$0 \in \Re^k$的邻域W, 存在单值*Lipschitz*连续映射$\phi : W \to \Re^k$, 其*Lipschitz*常数为γ, 满足

$$\phi(0) = z_0,\ \phi(\xi) \in \Sigma(\xi),\ \ \forall \xi \in W. \tag{8}$$

则对每一$\varepsilon > 0$, 存在x_0的邻域U_ε与z_0的邻域V_ε, 以及一单值映射$\sigma : U_\varepsilon \to V_\varepsilon$满足

$$\sigma(x_0) = z_0,\ \sigma(x) \in S(x),\ \ \forall x \in U_\varepsilon, \tag{9}$$

且映射σ在U_ε上是*Lipschitz*连续的, *Lipschitz*常数为$(\gamma + \varepsilon)L$, 其中L由(7)定义.

(b) 如果还有,存在z_0的一邻域V满足

$$\phi(\xi) = \Sigma(\xi) \cap V, \ \forall \xi \in W, \tag{10}$$

则

$$\sigma(x) = S(x) \cap V_\varepsilon, \ \forall x \in U_\varepsilon. \tag{11}$$

证明 先证明(a). 对任意固定的$\varepsilon > 0$, 选取$\delta = \delta(\varepsilon) > 0$, $\rho = \rho(\varepsilon) > 0$与$x_0$的一邻域$U_\varepsilon$, 满足对于$V_\varepsilon = z_0 + \rho\mathbf{B}$, 有

$$\begin{aligned}
&\gamma\delta < \varepsilon/(\gamma+\varepsilon), \\
&r(x,z) \in W, && \forall (x,z) \in U_\varepsilon \times V_\varepsilon, \\
&\|\mathcal{J}_z C(x_0,z_0) - \mathcal{J}_z C(x,z)\| \leq \delta, && \forall (x,z) \in U_\varepsilon \times V_\varepsilon, \\
&\|C(x_0,z_0) - C(x,z)\| \leq (1-\gamma\delta)\rho/\gamma, && \forall x \in U_\varepsilon.
\end{aligned} \tag{12}$$

$\sigma(\bar{x}) = ?$

把(a)的证明分成两部分: (i) 构造σ; (ii) 验证σ 的Lipschitz 连续性.

对每一固定的$\bar{x} \in U_\varepsilon$, 定义映射$\Phi_{\bar{x}} : \Re^k \to \Re^k$,

$$\Phi_{\bar{x}}(\cdot) := \phi(r(\bar{x}, \cdot)). \tag{13}$$

$r(\bar{x}, z) = C(x_0, z_0) + J_z C(x_0, z_0)(z - z_0) - C(\bar{x}, z).$

下面我们证明

$$\Phi_{\bar{x}} \text{ 是 } V_\varepsilon \text{ 上的一压缩映射, 它把} V_\varepsilon \text{映到} V_\varepsilon. \tag{14}$$

如果上述结论成立, 则由Banach不动点定理可得存在$\bar{z} \in V_\varepsilon$满足

$$\bar{z} = \Phi_{\bar{x}}(\bar{z}) = \phi(r(\bar{x}, \bar{z})).$$

$\sigma(\bar{x}) := \bar{z}$

于是根据(8),

$$\overline{z} \in \Sigma(r(\overline{x}, \overline{z})).$$

由命题2.1可得$\overline{z} \in S(\overline{x})$, 因为$\overline{x}$是$U_\varepsilon$中的任意点, 定义在$U_\varepsilon$上的映射

$$\sigma : x \to z \in S(x)$$

$\sigma(\overline{x}) = \overline{z}$

是存在的. 由

$$\Phi_{x_0}(z_0) = \phi(r(x_0, z_0)) = \phi(0) = z_0$$

可得$\sigma(x_0) = z_0$, 这证得(9). 下面只需验证(14)式.
为验证$\Phi_{\overline{x}}$的压缩性质, 对$z_1, z_2 \in V_\varepsilon$, 由$W$上定义的$\phi$的Lipschitz连续性质可得

$$\begin{aligned}&\|\Phi_{\overline{x}}(z_1) - \Phi_{\overline{x}}(z_2)\| \le \gamma\|r(\overline{x}, z_1) - r(\overline{x}, z_2)\| \\ &\le \gamma \cdot \sup\{\|\mathcal{J}_z r(\overline{x}, (1-\mu)z_1 + \mu z_2)\| : \mu \in (0,1)\} \cdot \|z_1 - z_2\|.\end{aligned}$$

由于$\mathcal{J}_z r(\overline{x}, z) = \mathcal{J}_z C(x_0, z_0) - \mathcal{J}_z C(\overline{x}, z)$, 由 (12)可得

$$\|\Phi_{\overline{x}}(z_1) - \Phi_{\overline{x}}(z_2)\| \leq \gamma\delta\|z_1 - z_2\|, \ \forall z_1, z_2 \in V_\varepsilon. \tag{15}$$

由δ的选取有$\gamma\delta < 1$, $\Phi_{\overline{x}}$实际上是一压缩映射. 进一步,

$$\begin{aligned} \|\Phi_{\overline{x}}(z_0) - z_0\| &= \|\phi(r(\overline{x}, z_0)) - \phi(0)\| \\ &\leq \gamma\|r(\overline{x}, z_0) - 0\| \\ &= \gamma\|C(x_0, z_0) - C(\overline{x}, z_0)\| \\ &\leq (1 - \gamma\delta)\rho. \end{aligned}$$

这意味着对于$z \in V_\varepsilon(= z_0 + \rho\mathbf{B})$,

$$\begin{aligned} \|\Phi_{\overline{x}}(z) - z_0\| &\leq \|\Phi_{\overline{x}}(z) - \Phi_{\overline{x}}(z_0)\| + \|\Phi_{\overline{x}}(z_0) - z_0\| \\ &\leq \gamma\delta\|z - z_0\| + (1 - \gamma\delta)\rho \leq \rho, \end{aligned} \tag{16}$$

即$\Phi_{\overline{x}}$映V_ε 到自身.

不等式(15)与(16) 表明，可以用Banach不动点定理，从而保证映射σ的存在性.
现在证明σ在U_ε上是Lipschitz连续的，Lipschitz常数是$(\gamma+\varepsilon)L$. 不妨设$U_\varepsilon \times V_\varepsilon \subset \widetilde{U} \times \widetilde{V}$，其中$\widetilde{U}, \widetilde{V}$由(7)定义，则对任意$x_1, x_2 \in U_\varepsilon$，

$$\begin{aligned}\|\sigma(x_1)-\sigma(x_2)\| &= \|\Phi_{x_1}(\sigma(x_1))-\Phi_{x_2}(\sigma(x_2))\| \\ &\le \|\Phi_{x_1}(\sigma(x_1))-\Phi_{x_1}(\sigma(x_2))\| + \|\Phi_{x_1}(\sigma(x_2))-\Phi_{x_2}(\sigma(x_2))\|.\end{aligned}$$

由(15)可得

$$\|\Phi_{x_1}(\sigma(x_1))-\Phi_{x_1}(\sigma(x_2))\| \le \gamma\delta\|\sigma(x_1)-\sigma(x_2)\|.$$

由ϕ的Lipschitz连续性可得

$$\begin{aligned}\|\Phi_{x_1}(\sigma(x_2))-\Phi_{x_2}(\sigma(x_2))\| &= \|\phi(r(x_1,\sigma(x_2)))-\phi(r(x_2,\sigma(x_2)))\| \\ &\le \gamma\|C(x_1,\sigma(x_2))-C(x_2,\sigma(x_2))\|.\end{aligned}$$

结合这些估计和(7)得到

$$\begin{aligned}\|\sigma(x_1)-\sigma(x_2)\| &\leq \gamma\delta\|\sigma(x_1)-\sigma(x_2)\| \\ &\quad +\gamma\|C(x_1,\sigma(x_2))-C(x_2,\sigma(x_2))\| \\ &\leq \gamma\delta\|\sigma(x_1)-\sigma(x_2)\|+\gamma L\|x_1-x_2\|,\end{aligned}$$

由此可推出

$$\|\sigma(x_1)-\sigma(x_2)\| \leq \frac{\gamma L}{1-\gamma\delta}\|x_1-x_2\| < (\gamma+\varepsilon)L\|x_1-x_2\|,$$

即σ在U_ε上是Lipschitz连续的.

再来证明(b). 如果有必要, 可以选择(12)中的ρ充分小, 所以可以假设$V_\varepsilon \subset V$. 现在固定$x \in U_\varepsilon$, 令z是从$S(x) \cap V_\varepsilon$中任意选取的元素. 为证明(11), 只需证明$z = \sigma(x)$. 根据命题2.1有$z \in \Sigma(r(x,z)) \cap V_\varepsilon$. 由(12)可得$r(x,z) \in W$, 于是由假设(10)和定义式(13)有

$$z = \phi(r(x,z)) = \Phi_x(z).$$

因为$\Phi_x(\cdot)$在V_ε仅有一个不动点, z必是由(a)确定的唯一的不动点$\sigma(x)$, 这证得

$$\sigma(x) = S(x) \cap V_\varepsilon, \ \forall x \in U_\varepsilon.$$

■

再归纳

定义 2.3

$\delta \in f(\bar{x}, \bar{p}) + D_x f(\bar{x}, \bar{p})(x-\bar{x}) + F(x)$

$\downarrow$ $\phi(\delta): {}_{\varepsilon_1}\mathbb{B} \to \mathbb{B}_{\varepsilon_2}(\bar{x})$

考虑广义方程

$$“0 \in f(x, p) + F(x),”$$

其中$F : \mathcal{X} \rightrightarrows \mathcal{Y}$是一集值映射. 定义

$$G(x) = f(\bar{x}, \bar{p}) + D_x f(\bar{x}, \bar{p})(x - \bar{x}) + F(x).$$

如果G^{-1}是从$0 \in \mathcal{Y}$的一个邻域到$\bar{x}$的一邻域的Lipschitz连续映射,则称广义方程在$(\bar{x}, \bar{p})$处是强正则的.

解映射

由参数广义方程定义的解映射

$$S(x) = \{z \in \mathbb{R}^k : 0 \in C(x, z) + N_Q(z)\}, \qquad (17)$$

其中$C : \mathcal{A} \times \mathbb{R}^k \to \mathbb{R}^k$ 是一连续可微映射, $\mathcal{A} \subset \mathbb{R}^n$是一开集合, $Q \subset \mathbb{R}^k$是一非空闭凸集合.

定理 2.4

给定$(x_0, z_0) \in \operatorname{gph} S$, $x_0 \in \mathcal{A}$. 如果(x_0, z_0)是系统(17)的强正则解,则存在x_0的一个邻域$\mathcal{O}$和z_0的一个邻域V, 满足在$\mathcal{O}$上存在一个单值的*Lipschitz*连续映射$\sigma : \mathcal{O} \to V$使得

$$\sigma(x_0) = z_0,\ \sigma(x) \in S(x),\ \ \forall x \in \mathcal{O}, \tag{18}$$

或者使得

$$\sigma(x) = S(x) \cap V,\ \ \forall x \in \mathcal{O}. \tag{19}$$

一般集值映射的强正则性

严格图导数准则

- 针对集值映射的强正则性, Klatte and Kummer[13][8]利用严格图导数的性质给出了强正则性的刻画.
- 称集值映射$F : X \rightrightarrows Y$在$(\bar{x}, \bar{y})$处是强正则的, 若其逆映射$F^{-1}$在$(\bar{y}, \bar{x})$处具有Aubin性质, 且还分别存在$\bar{x}$的邻域$U$, $\bar{y}$的邻域V, 使得对$y \in V$, 有$U \cap F^{-1}(y)$是单值的.
- 称严格图导数$D_*F(\bar{x}|\bar{y})$是单射, 若有

$$0 \in D_*F(\bar{x}|\bar{y})(u) \Rightarrow u = 0.$$

$V \in \mathcal{N}(\bar{y})$
$U \in \mathcal{N}(\bar{x})$

$$\underbrace{F^{-1}(y) \cap U}_{\text{单值}} \subseteq F^{-1}(y') + \kappa \|y' - y\| \mathbb{B}_Y, \quad \forall y', y \in V$$

[8]Klatte D and Kummer B. *Nonsmooth Equations in Optimization*. Kluwer Academic, 2002.

$0 \in D_* F(\bar{z}|\bar{y})(u) \Rightarrow u = 0$

定理 2.5

[13, Lemma 3.1] (强正则性的严格图导数准则) 设集值映射$F : X \rightrightarrows Y$(赋范空间), $\bar{z} = (\bar{x}, \bar{y}) \in \mathrm{gph} F$. 则有

(a) 若F在$\bar{z}$处是强正则的, 那么$D_* F(\bar{z})$是单射;

(b) 若$X = \Re^n$, 则F在$\bar{z}$处是强正则的充要条件是$D_* F(\bar{z})$是单射且F^{-1} 在$(\bar{y}, \bar{x})$处是*Lipschitz*下半连续的.

S is called Lipschitz l.s.c. at (x, y) with rank L

$$\mathrm{dist}\,(y, S(x')) \le L d_{\mathcal{X}}(x, x')$$

$S : X \rightrightarrows Y$

for all x' in some neighborhood V of x.

证明

反证法. (a) 假设$D_*F(\bar z)$[9] 不是单射, 则$\exists u \neq 0$, 使得$0 \in D_*F(\bar z)(u)$. 由定义知, 存在$\eta^k \in F(x^k + t_k u^k)$, $t_k \downarrow 0$, $\mathrm{gph} F \ni (x^k, y^k) \to \bar z$, $u^k \to u$, 有$v^k = (\eta^k - y^k)/t_k \to 0$.

令$\xi^k = x^k + t_k u^k$, 则

$\eta^k \in S(\xi^k) \Leftrightarrow (\xi^k, \eta^k) \in \mathrm{gph}\, S$

$$\text{存在}(x^k, y^k), (\xi^k, \eta^k) \xrightarrow{\mathrm{gph} F} \bar z, \text{使得} \frac{d(\xi^k, x^k)}{d(\eta^k, y^k)} = \|u^k\| \|v^k\|^{-1} \to \infty, \tag{20}$$

$\xi^k \in F^{-1}(\eta^k)$, $x^k \in F^{-1}(y^k)$

这与F在$\bar z$处是强正则的矛盾.

---

[9] 回顾定义

$\eta^k \in F(x^k + t_k u^k)$, $\frac{1}{t_k}(\eta^k - y^k) \to v$

$$D_*F(\bar x|\bar y)(u) := \Big\{ v \mid \exists \tau^\nu \searrow 0, (x^\nu, y^\nu) \overset{\mathrm{gph}\, F}{\to} (\bar x, \bar y), u^\nu \to u,$$
$$\text{with } v^\nu \in [F(x^\nu + \tau^\nu u^\nu) - y^\nu]/\tau^\nu, v^\nu \to v \Big\}$$

(b) 设$X = \Re^n$. 假设F在$\bar{z}$处不是强正则的, 这等价于

$$存在y^k \to \bar{y}, 使得d(\bar{x}, F^{-1}(y^k)) > kd(y^k, \bar{y}). \tag{21}$$

或存在$(x^k, y^k) \in \mathrm{gph}\, F$, $(\xi^k, \eta^k) \in \mathrm{gph}\, F$它们都收敛到$\bar{z}$,满足

$$\frac{d(\xi^k, x^k)}{d(\eta^k, y^k)} > k. \tag{22}$$

其中, (21)意味着F^{-1}在$(\bar{y}, \bar{x})$处不是Lipschitz下半连续的.
若(22)成立,(特别地, 当$F^{-1}(y^k)$在$\bar{x}$附近是多值时),
设$t_k = \|\xi^k - x^k\|$ 且$u^k = (\xi^k - x^k)/t_k$. (22) 等价于满足以下条件的这些序列的存在性：

$0 \in D_*F(\bar{x}|\bar{y})(u)$

$$\eta^k \in F(x^k + t_k u^k), t_k \downarrow 0, \mathrm{gph} F \ni (x^k, y^k) \to \bar{z},$$

$u \neq 0$

$$\|u^k\| = 1, v := \lim(\eta^k - y^k)/t_k = 0. \Leftarrow \frac{d(\xi^k, y^k)}{d(\xi^k, x^k)} < \frac{1}{k}. \checkmark$$

由于$X=\Re^n$, 则$\exists u\neq 0$, 使得$u^k\to u$, 则有$\exists u\neq 0$, $v=\lim(\eta^k-y^k)/t_k=0$, 其中$\eta^k\in F(x^k+t_ku^k)$, $t_k\downarrow 0$, $\mathrm{gph}F\ni(x^k,y^k)\to\bar z$, $u^k\to u$,即$\exists u\neq 0$, 使$0\in D_*F(\bar z)(u)$, 与单射条件矛盾.

反之, 假设$D_*F(\bar z)$不是单射或F^{-1}在$(\bar y,\bar x)$处不是Lipschitz下半连续的, 由(a)知第一种情况与F在$\bar z$处是强正则的矛盾, 而第二种情况也与F在$\bar z$处是强正则的矛盾. $\blacksquare$

设X, Y是两个有限维的Hilbert空间, $(\bar{x},\bar{y}) \in X \times Y$, 记

$$\pi_x \partial H(\bar{x},\bar{y}) = \partial H(\bar{x},\bar{y}) \text{ 到空间 } X\text{上的投影} .$$

下面的引论是需要的

引理 2.1

设 $H : X \times Y \to X$ 是$(\bar{x},\bar{y}) \in X \times Y$的某一开邻域上的局部*Lipschitz*连续函数, $H(\bar{x},\bar{y}) = 0$. 如果$\pi_x \partial H(\bar{x},\bar{y})$中的每一元素均是非奇异的, 则存在$\bar{y}$的一开邻域 $\mathcal{O}_Y$与一局部*Lipschitz*连续函数 $x(\cdot) : \mathcal{O}_Y \to X$ 满足$x(\bar{y}) = \bar{x}$且对每一$y \in \mathcal{O}_Y$,

$$H(x(y), y) = 0 .$$

进一步, 如果H在$(\bar{x},\bar{y})$的开邻域中的每一点均是(强)半光滑的, 则$x(\cdot)$在$\mathcal{O}_Y$中的每一点均是(强)半光滑的.

证明

结合Clarke关于局部Lipschitz连续函数的隐函数定理[4, Section 7.1][10]可直接得到前半部分结论成立. 后半部分的证明由 [25, Corollary 2.1][11]和[27, Lemma 2][12]可知,如果H在$(\bar{x}, \bar{y})$的开邻域中的每一点均是(强)半光滑的, 则$x(\cdot)$在$\mathcal{O}_Y$中的每一点均是(强)半光滑的. ∎

[10] Clarke F H. *Optimization and Nonsmooth Analysis*. New York: John Wiley and Sons, 1983.

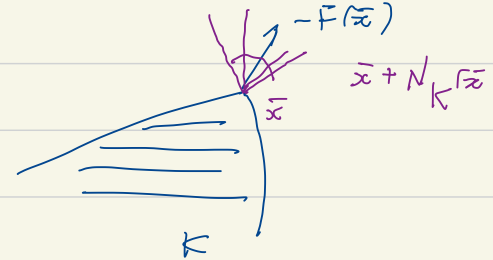
[11] Sun D F. *A further result on an implicit function theorem for locally Lipschitz functions*. Operations Research Letters, 2001, **28**: 193-198.

[12] Kummer, B. *Newton's Method Based on Generalized Derivatives for Nonsmooth Functions: Convergence Analysis*. In: Oettli, W., Pallaschke D. eds., Lecture Notes in Economics and Mathematical Systems 382; Advances in Optimization. Springer, Berlin, 1992, 171-194.

(VI) 求 $\bar{x}(\delta)$

$$\langle F(\bar{x})+\delta,\ x-\bar{x}(\delta)\rangle \geq 0,\quad \forall x \in K$$

闭凸集



$$\Leftrightarrow \quad 0 \in F(\bar{x}(\delta)) + N_K(\bar{x}(\delta))$$

$$\Leftrightarrow \quad \bar{x}(\delta) - \Pi_K(\bar{x}(\delta) - F(\bar{x}(\delta))) = 0$$

非光滑方程问题

$$x - \Pi_K(x - F(x)) = 0$$

对应扰动

$$x - \underbrace{\Pi_K}(x - F(x) - \delta) = 0$$

$$\Rightarrow x(\delta)$$

Clarke 隐函数定理

应用场景

变分不等式VI(G, K):求$x \in K$ 满足

$$\langle G(x), z - x \rangle \geq 0 \quad \forall z \in K.$$

它等价于

$$-G(x) \in N_K(x).$$

扰动问题

$$-G(x) - y \in N_K(x).$$

用Natural mapping表达成非光滑方程:

$$H(x, y) = x - \Pi_K(x - G(x) - y) = 0.$$

Clarke广义Jacobian

- H的 Clarke广义Jacobian

$$\partial H(x,y) = \left\{ [\mathcal{I} - (\mathcal{I} - DG(x)^*)V \; V] : V \in \partial \Pi_K(x - G(x) - y) \right\}.$$

- 到$\mathcal{X}$上的投影

$$\pi_x \partial H(x,y) = \left\{ \mathcal{I} - (\mathcal{I} - DG(x)^*)V : V \in \partial \Pi_K(x - G(x) - y) \right\}.$$

- 很多情况下Π_K是半光滑的,比如$K = P$,P是凸的多面体集合,$K = \mathbb{S}^p$, $K = Q_{m+1}$, $K = \mathrm{epi}\,\|\cdot\|$, $\|\cdot\|_2$是谱范数,$\|\cdot\|_*$是核范数.

定理 2.6

[24, Theorem 9.54] (单值局部化). 设$S : \Re^n \rightrightarrows \Re^m$, $\bar{u} \in S(\bar{x})$. 设S以下述含义在$\bar{x}$ 相对于$\bar{u}$是局部内半连续的, 即存在邻域$V \in \mathcal{N}(\bar{x})$与$W \in \mathcal{N}(\bar{u})$, 满足对任何$x \in V$与$\varepsilon > 0$, 存在$\delta > 0$,

$$\left.\begin{array}{l} S(x) \cap W \subset S(x') + \varepsilon \boldsymbol{B} \\ S(x') \cap W \subset S(x) + \varepsilon \boldsymbol{B} \end{array}\right\} \text{当 } x' \in V \cap \boldsymbol{B}(x, \delta).$$

(a) S在$\bar{x}$处相对于$\bar{u}$具有一*Lipschitz* 连续的单值局部化T当且仅当$D_* S(\bar{x}|\bar{u})(0) = \{0\}$.

(b) 如果S是凸值的, 则S^{-1}在$\bar{u}$处相对于$\bar{x}$有*Lipschitz*连续的单值局部化T当且仅当$D_* S(\bar{x}|\bar{u})^{-1}(0) = \{0\}$且$m = n$.

推论 2.1

[24, Corollary 9.55] (单值*Lipschitz*函数的可逆性).
令$\mathcal{O} \subset \Re^n$是一开集合, $F : \mathcal{O} \to \Re^n$是一连续映射. 对于$\bar{x} \in \mathcal{O}$, F^{-1}在$\bar{u} = F(\bar{x})$处具有一*Lipschitz*连续的单值局部化的充分必要条件是F满足非奇异严格导数条件:

$$D_* F(\bar{x})(w) = 0 \Longrightarrow w = 0.$$

✓

证明. 本推论是将集值映射S替换为单值映射F后定理2.6(b)的具体化. ∎

以下的内容取自Kummer (1991)[14].[13]
下面的定义可用于刻画局部Lipschitz可逆性质.

定义 2.4

[14, Definition 1.3]设 $F:\Re^n\to\Re^m$ 是一个连续函数, $x\in\Re^n$, 定义 $\Delta F(x)$为

$$\Delta F(x)=\left\{ z \;\middle|\; \begin{array}{l} \exists x^\nu\to x, y^\nu\to x, x^\nu\neq y^\nu \\ \text{使得 } \dfrac{F(y^\nu)-F(x^\nu)}{\|y^\nu-x^\nu\|}\to z \end{array} \right\}.$$

[13]Kummer B. *Lipschitzian inverse functions, directional derivatives, and applications in $C^{1,1}$-optimization*. Journal of Optimization Theory and Applications, 1991, **70**: 559-580.

引理 2.2

[14, Lemma 2.1] 连续函数 $F:\Re^n\to\Re^n$在x附近是局部$Lipschitz$可逆的当且仅当 $0\notin\Delta F(x)$.

证明. ($\Rightarrow$) 利用反证法. 假设连续函数F在x附近是局部Lipschitz可逆的, 但 $0\in\Delta F(x)$, 则存在
$x^\nu\to x, y^\nu\to x\,, x^\nu\neq y^\nu$使得

$$[F(y^\nu)-F(x^\nu)]/\|y^\nu-x^\nu\|\to 0$$

成立. 而由F^{-1}在$F(x)$处局部Lipschitz连续可得, 存在正数κ使得

$y^\nu - x^\nu$

$$\|F^{-1}(F(y^\nu))-F^{-1}(F(x^\nu))\|\le\kappa\|F(y^\nu)-F(x^\nu)\|$$

成立. 这产生了矛盾.

$(\Leftarrow)$ 若$0 \notin \Delta F(x)$, 则存在正数ε和κ使得

$$\|F(x') - F(x'')\| \geq \kappa \|x' - x''\|, \; \forall \; x', x'' \in B(x, \varepsilon). \tag{23}$$

由(23)知F是一单射, 再注意它是连续的, 则对于开集合$\mathcal{O} \subset \Re^n$, 有$F(\mathcal{O})$是开集合. 于是存在$\delta > 0$使得$\mathbf{B}(F(x), \delta) \subset F(\mathbf{B}(x, \varepsilon))$. 所以由(23)可得$F(y) = z, y \in \mathbf{B}(x, \varepsilon)$存在唯一解$y = F^{-1}(z)$且 F^{-1}在$\mathbf{B}(F(x), \delta)$上是Lipschitz连续的. ■

引理 2.3

[14, Lemma 2.2][14] 若 $F \in \mathcal{C}^{0,1}(\Re^n, \Re^m)$, 则

$$\Delta F(x) = \bigcup_{\|u\|=1} D_* F(x)(u).$$

证明. 由定义可知, $\bigcup_{\|u\|=1} D_* F(x)(u) \subset \Delta F(x)$是显然的. 现证明相反的包含关系.

$z \in D_* F(\bar{x})(w) \Leftrightarrow \exists \tau^\nu \searrow 0,\ x^\nu \to \bar{x},\ w^\nu \to w,\ \frac{F(x^\nu + \tau^\nu w^\nu) - F(x^\nu)}{\tau^\nu} \to z$ ✓

Lipschitz Continuity

[14] Note that if F is strictly continuous, one then has the simpler formula

$$D_* F(\bar{x})(w) = \left\{ z \middle| \exists \tau^\nu \searrow 0, x^\nu \to \bar{x} \text{ with } \Delta_{\tau^\nu} F(x^\nu)(w) \to z \right\}.$$

任取 $z \in \Delta F(x)$, 则存在 $x^\nu \to x, y^\nu \to x$, $x^\nu \neq y^\nu$ 使得

$$z^\nu = [F(y^\nu) - F(x^\nu)]/\|y^\nu - x^\nu\| \to z$$

成立. 令 $\lambda^\nu = \|y^\nu - x^\nu\|, u^\nu = (y^\nu - x^\nu)/\lambda^\nu$, 则 $\{u^\nu\}$是有界数列, 不妨设 $u^\nu \to u$, 则 $\|u\| = 1$. 定义

$$v^\nu = (F(x^\nu + \lambda^\nu u) - F(x^\nu))/\lambda^\nu,$$

$F(x^\nu + \lambda^\nu u^\nu)$

则对充分大的 ν, 存在正数 κ 使得

$z^\nu = \frac{1}{\lambda^\nu}[F(x^\nu + \lambda^\nu u^\nu) - F(x^\nu)]$

$$\begin{aligned} \|z^\nu - v^\nu\| &= \|F(y^\nu) - F(x^\nu + \lambda^\nu u)\|/\lambda^\nu \\ &= \|F(x^\nu + \lambda^\nu u^\nu) - F(x^\nu + \lambda^\nu u)\|/\lambda^\nu \\ &\leq \kappa\|\lambda^\nu(u^\nu - u)\|/\lambda^\nu \end{aligned}$$

$\{u^\nu\}$ 聚点 $u \neq 0$.

成立. 所以 $\lim_{\nu\to\infty} z^\nu = z = \lim_{\nu\to\infty} v^\nu \in D_*F(x)(u)$. ■

基于上述引理可以得到逆函数定理.

定理 2.7

[14, Theorem 1.1](*Kummer*逆函数定理). 设函数 $F : \mathcal{O} \subset \mathcal{X} \to \mathcal{X}$ 在 $\bar{x} \in \mathcal{O}$附近是局部*Lipschitz*连续的. 则 F在 $\bar{x}$ 附近是*Lipschitz*同胚的当且仅当下述非奇异条件成立:

$$0 \notin D_* F(\bar{x})(u), \quad \forall\, 0 \neq u \in X.$$

计算严格图导数的问题

$0 \notin \Delta F(\bar{x}) \Longleftrightarrow F^{-1}$ Lipschitz continuous

证明

由引理2.2和引理2.3可知，F 在$\bar{x}$附近是Lipschitz同胚的当且仅当

$$0 \notin \underbrace{\bigcup_{\|u\|=1} D_* F(\bar{x})(u)}_{\Delta F(\bar{x})}.$$

而由$D_* F(x)$的正齐次性可得 （引理2.3）

$$0 \notin \bigcup_{\|u\|=1} D_* F(\bar{x})(u)$$

等价于 $0 \notin D_* F(\bar{x})(u), \forall\, 0 \neq u \in X.$ ■

孤立平稳性的图导数准则

结合[15, Proposition 4.1] [15]及[11, Proposition 2.1][16], 给出集值映射孤立平稳性的图导数准则, 即用集值映射的图导数刻画孤立平稳性.

$S(u) \cap W \subseteq \{\bar{y}\} + \kappa \|u - \bar{u}\| \mathbb{B}$ $\quad W \in \mathcal{N}(\bar{y})$, $u \in V$, $V \in \mathcal{N}(\bar{u})$

定理 2.8

(孤立平稳性的图导数准则) 设 X, Y是两个有限维的*Hilbert*空间. 设集值映射$F : X \rightrightarrows Y$. 对$(\bar{x}, \bar{y}) \in \mathrm{gph} F$, F在$\bar{x}$处关于$\bar{y}$孤立平稳的充要条件是$\{0\} = DF(\bar{x}|\bar{y})(0)$.

$$DF(\bar{x}|\bar{y})(w) = \limsup_{\substack{\tau \downarrow 0 \\ w' \to w}} \frac{F(\bar{x} + \tau w') - \bar{y}}{\tau}$$

[15] Levy A B. *Implicit multifunction theorems for the sensitivity analysis of variational conditions*. Math. Program., 1996, **74**: 333-350.

[16] King A and Rockafellar R T. *Sensitivity analysis for nonsmooth generalized equations*. Math Program, 1992, **55**: 341 - 364.

证明

要证 $v=0$

必要性.[17]考虑任一$v \in DF(\bar{x}|\bar{y})(0)$, 则存在序列$v^k \to v$, $u^k \to 0$, $t_k \downarrow 0$, 满足对$\forall k$, 有$\bar{y} + t_k v^k \in F(\bar{x} + t_k u^k)$. 因为$F$在$\bar{x}$处关于$\bar{y}$孤立平稳的, 有$\bar{x}$的邻域$V$, $\bar{y}$ 的邻域W, 常数$\kappa > 0$满足$F(x) \cap W \subseteq \{\bar{y}\} + \kappa\|x - \bar{x}\|\mathbb{B}_Y$, $\forall x \in V$. 对充分大的k, 有$\bar{y} + t_k v^k \in \bar{y} + \kappa\|t_k u^k\|\mathbb{B}_Y$, 即$v^k$ 包含在半径为$\kappa\|u^k\|$ 的球中, 由于$\{u^k\} \to 0$, 则v必为0.

---

[17] 回顾

$$
\begin{aligned}
DS(\bar{x}|\bar{u})(\bar{w}) &= \limsup_{\tau \searrow 0, w \to \bar{w}} \frac{S(\bar{x} + \tau w) - \bar{u}}{\tau} \\
&= \left\{ v : \exists t^\nu \searrow 0, w^\nu \to \bar{w}, \exists u^\nu \in S(\bar{x} + \tau^\nu w^\nu), \underbrace{\frac{u^\nu - \bar{u}}{\tau^\nu}}_{} \overset{= v^k}{\to} v \right\}.
\end{aligned}
$$

$DF(\bar{x}|\bar{y})(0)=\{0\}$

充分性. 反证法. 假设F在$\bar{x}$处关于$\bar{y}$不是孤立平稳的, 则存在序列$x^k \to \bar{x}$, $\exists y^k \in F(x^k)$, 使得$y^k \notin \{\bar{y}\} + k\|x^k - \bar{x}\|\mathbb{B}_Y$, 则有$\|y^k - \bar{y}\| > k\|x^k - \bar{x}\|$. 令$t_k = \|y^k - \bar{y}\|$, $v^k = (y^k - \bar{y})/t_k$, 则$\|v^k\| = 1$, 由于$Y$有限维, $\exists v \neq 0$, 使得$v^k \to v$. 令$u^k = (x^k - \bar{x})/t_k$,则

$$\|u^k\| = \|x^k - \bar{x}\|/\|y^k - \bar{y}\| < 1/k \to 0,$$

$v \in DF(\bar{x}|\bar{y})(0)$

即存在$t_k \downarrow 0$, $(u^k, v^k) \to (0, v)$满足$\bar{y} + t_k v^k \in F(\bar{x} + t_k u^k)$, 这意味着$0 \neq v \in DF(\bar{x}|\bar{y})(0)$, 与条件矛盾. ■

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